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On Multi-Head Finite Automata†

Abstract: Let $\mathfrak{M}_n$ be the class of languages defined by n-head finite automata. The Boolean and Kleene closure properties of $\mathfrak{M}_n$ are investigated, and a relationship between $\mathfrak{M}_n$ and the class of sets of n-tuples of tapes defined by n-tape finite automata is established. It is shown that the classes $\mathfrak{M}_i$ form a hierarchy; and that, moreover, for all n, there is a context-free language (CFL) in $\mathfrak{M}_{n+1} - \mathfrak{M}_n$. It is further shown that there is a CFL which is in no $\mathfrak{M}_n$ for any integer n. Finally, several decision properties of the multi-head languages are established.

Introduction

When faced with the task of processing an artificial language, such as a programming language, on a computer, one often employs an algorithm that simulates the action of an automaton. For example, a pushdown algorithm (Burks et al.[1]), which simulates the performance of a pushdown automaton (Evey[2]), is frequently used in the processing of arithmetic expressions or statements in the propositional calculus. Usually, however, there is no unique algorithm, nor even a unique class of algorithms, for processing a given language. One must, therefore, choose that algorithm which will be most efficient for performing the job at hand. It follows that one major problem in the area of "linguistic" automata theory is that of determining the relative computational powers and degrees of efficiency of different models of syntactic analyzers.

If an automaton requires no work tape in its computations (i.e., is a *nonwriting* model) then, in the simulation of this model for the processing of a language, one need not establish and maintain the (often costly) list structures that are usually employed to simulate the work tape of an automaton. The automaton is, in a sense, more efficient than any *writing* model (such as a pushdown automaton) that will do the same job.

In view of the computational weakness of the model of a finite automaton we are led, from the above motivation, to the investigation of extensions of this basic model which preserve its nonwriting character. Rabin and Scott[3] and the author[4,5] have investigated several nonwriting extensions of finite automata. In this paper we extend the study reported in Ref. 5 of finite automata with several READ heads on their input tape. We designate the class of languages which can be analyzed by such automata as *multi-head languages*.

After defining the model to be studied, we turn to an investigation of the closure properties of the multi-head languages under the Boolean (union, intersection, complementation) and Kleene (concatenation, closure, and reversal) operations. We establish the relationship of the multi-head languages to the finite-state, context-free, and context-sensitive languages; and we generalize the adage, "two heads are better than one," to "$n + 1$ heads are better than n." Finally we consider several decision problems associated with multi-head finite automata.

Definitions and notation

We assume familiarity with the concepts of alphabet, tape, set of tapes, and the Boolean and Kleene operations on sets of tapes. (See, for example, Rabin and Scott[3]). We let $\#(S)$ denote the cardinality of the set S.

An n-head finite automaton (n-FA) operates much as does a finite automaton in the sense of Rabin and Scott, the main difference being in that an n-FA has n read-only scanning devices on its input tape. Our formalism differs in two respects from that of Rabin and Scott. First, for reasons that are more aesthetic than scientific, we designate two internal configurations of our n-FA to be halting states; every computation is ended by entering one of these states. Secondly, we give our n-FA's information about when they have reached the end of the input tape. In fact, we view an input tape t as being placed on an input channel in the form $t\$$, where $\$$ is a special end-tape symbol. While

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neither the special halting states nor the technical symbol alters the computational power of a finite automaton (see Remark 1), one can easily convince himself that the computational power of an n-FA ($n > 1$) is increased by the addition of an end-tape symbol.

Definition 1. *An n-head* ($n > 0$) *finite automaton, n*-FA, *is an 8-tuple* $(K, s_A, s_R, \Sigma, \$, h, M, s_0)$ *where*

(1) *K is a finite set (of states) which is the union of the* $2^n - 1$ *non-empty sets* $K' \times (2^{\{1,\cdots,n\}} - \{\{1,\cdots,n\}\})$; (Subsequent notation is simplified.)†

(2) s_A *and* s_R *are special symbols, not in K (the accept and reject states, respectively)*;

(3) Σ *is an alphabet (of input symbols)*;

(4) \$ *is a symbol not in* Σ *(the end-tape symbol)*;

(5) *h (the head-selector function) is a function from K into* $\{1, \cdots, n\}$ *which maps*
$h : K' \times \{S\} \to \{1, \cdots, n\} - S$;
(S *in* $2^{\{1,\cdots,n\}} - \{\{1, \cdots, n\}\}$);

(6) *M (the next state function) is a function on* $K \times (\Sigma \cup \{\$\})$:
(a) $M : (K' \times \{S\}) \times \Sigma \to K' \times \{S\}$,
(b) *For q in* $K' \times \{S\}$,
$M(q, \$)$ *is in* $K' \times \{S \cup \{h(q)\}\}$ *if* $\#(S) < n - 1$,
$M(q, \$)$ *is in* $\{s_A, s_R\}$ *if* $\#(S) = n - 1$;

(7) s_0 *is in* $K' \times \{\phi\}$ *(the initial state).*

Since s_A, s_R, Σ, *and* \$ *will be understood, we shall usually denote an n*-FA *as* (K, M, h, s_0).

An n-FA consists of a control unit, which is capable of assuming a finite number of distinct internal configurations called *states*, and an input channel on which is a tape of the form $t\$$, where t is a tape over the alphabet Σ, and \$ is a symbol not in Σ, the end-tape symbol. Initially, the n-FA is in state s_0 with all n READ heads positioned on the leftmost symbol of t. (For simplicity we assume idealized heads which can coexist on the same square of the tape.) At each time unit, the head controlled by the current state (via the h function) reads the symbol on which it is now positioned and moves one square to the right. On the basis of the current state and the symbol scanned, the n-FA changes state. The n-FA halts upon entering state s_A or s_R which occurs after all n READ heads have scanned the end-tape symbol; the tape t is accepted or rejected according as the halting state is s_A or s_R, respectively.

We make the restriction‡ that once a READ head has scanned the end-tape symbol, no further READS are initiated with that READ head.

We formalize the concept of a computation by an n-FA.

Definition 2. *Let* $\mathfrak{A}$ *be an n*-FA *over* Σ *with state set K. If the tape t is on the tape channel of* $\mathfrak{A}$, *then the instantaneous description* (i.d.) *of* $\mathfrak{A}$ *at time k is the pair* $I_\mathfrak{A}(k) = \langle s, w\rangle$ *where*

1) *s is a member of K*;
2) *w is of the form* $w = x_1 m_{i_1} x_2 m_{i_2} \cdots m_{i_{n-1}} x_n m_{i_n} x_{n+1}$ *where* $x_1, \cdots, x_{n+1}$ *are (possibly null) tapes,* $\{i_1, \cdots, i_n\} = \{1, \cdots, n\}$, $x_1 x_2 \cdots x_n x_{n+1} = t\$$, *and the* m_{i_j} *are distinct symbols, not in* Σ, *which denote the positions of the* READ *heads.*

Let $L(t)$ denote the length of the tape t.

Definition 3. *Let* $\mathfrak{A} = (K, M, h, s_0)$ *be an n*-FA. *The computation of a tape t by* $\mathfrak{A}$ *is a sequence of* i.d.'s $I_\mathfrak{A}(0), \cdots, I_\mathfrak{A}(k)$ *where* $k = n(L(t) + 1)$ *such that:*

1) $I_\mathfrak{A}(0) = \langle s_0, m_1 m_2 \cdots m_n t\$\rangle$.
2) *For* $0 \le j < k$, *if* $I_\mathfrak{A}(j) = \langle s, x m_i u \sigma y\rangle$ *for* $h(s) = i$, *u a member of the Kleene closure*§ *of* $\{m_1, \cdots, m_{i-1}, m_{i+1}, \cdots, m_n\}$, *and* σ *in* $\Sigma \cup \{\$\}$, *then* $I_\mathfrak{A}(j + 1) = \langle M(s, \sigma), x u \sigma m_i y\rangle$.
3) $I_\mathfrak{A}(k) = \langle s, w\rangle$, *where* $w = t\$m_1 \cdots m_n$, *and s is in* $\{s_A, s_R\}$.

It is clear from the definition of an n-FA that such a k exists and is unique.

An n-FA $\mathfrak{A}$ *accepts* a tape t iff there is a computation $I_\mathfrak{A}(0), \cdots, I_\mathfrak{A}(k)$ of t such that the state of $\mathfrak{A}$ at time k is s_A. $\mathfrak{M}_n$ is the class of sets of tapes defined (accepted) by n-FA's.

Remark 1. A 1-FA is a finite automaton in the sense of Rabin and Scott. Thus $\mathfrak{M}_1$ is precisely the class of regular sets.

Chapter I. Characterization and closure properties

• *Characterization*

We begin this section by finding a characterization of $\mathfrak{M}_n$ in terms of the class of sets of n-tuples of tapes defined by deterministic n-tape automata.[4,5] We recall the definition of an n-tape automaton.

Definition 4. *An n-tape* ($n > 0$) *finite automaton, n*-DA, *is an 8-tuple* $(K, s_A, s_R, \Sigma, \$, f, M, s_0)$ *where*

(1) *K is a finite set (of states) which is the union of the* $2^n - 1$ *non-empty sets*
$$K' \times (2^{\{1,\cdots,n\}} - \{\{1, \cdots, n\}\});$$

(2) s_A *and* s_R *are special symbols, not in K (the accept and reject states, respectively)*;

(3) Σ *is an alphabet (of input symbols)*;

(4) \$ *is a symbol not in* Σ *(the end-tape symbol)*;

(5) *f (the tape-selector function) is a function from K into* $\{1, \cdots, n\}$ *which maps*
$$f : K' \times \{S\} \to \{1, \cdots, n\} - S,$$
for S in $(2^{\{1,\cdots,n\}} - \{\{1, \cdots, n\}\})$;

† We shall often write $K' \times S$ for $K' \times \{S\}$ when no confusion can occur.

‡ This restriction is formalized in the definition of the h function and the definition of the M function.

§ Subsequently we denote the Kleene closure of a set S by S^*.

(6) M (*the next state function*) *is a function on* $K \times (\Sigma \cup \{\$\})$:

(a) $M : (K' \times \{S\}) \times \Sigma \rightarrow K' \times \{S\}$,

(b) *for* q *in* $K' \times \{S\}$,

$$M(q, \$) \text{ is in } K' \times \{S \cup \{f(q)\}\} \quad \text{if } \#(S) < n - 1,$$

$$M(q, \$) \text{ is in } \{s_A, s_R\} \text{ if } \#(S) = n - 1;$$

(7) s_0 *is in* $K' \times \{\phi\}$ (*the initial state*).

As with an n-FA, we shall denote an n-DA as (K, M, f, s_0).

An n-DA works much as does an n-FA except that an n-DA has n input channels with one READ head per channel.

Definition 5. (a) *Let* $\mathfrak{A}$ *be an* n-*DA over* Σ *with state set K. If the n-tuple of tapes* $\langle t_1, \cdots, t_n \rangle$ *is on the tape channels of* $\mathfrak{A}$, *then the instantaneous description of A at time k is the* $(n+1)$-*tuple* $I_{\mathfrak{A}}(k) = \langle s, w_1, \cdots, w_n \rangle$ *where*

1) s *is a member of* K;
2) *for* $i = 1, \cdots, n$, w_i *is of the form* $w_i = x_i m y_i$ *where* x_i *and* y_i *are (possibly null) tapes,* $x_i y_i = t_i\$$, *and* m *is a symbol, not in* Σ, *which denotes the position of the* READ *head.*

(b) *If* $\mathfrak{A} = (K, M, f, s_0)$, *then the computation of an n-tuple of tapes* $\langle t_1, \cdots, t_n \rangle$ *by* $\mathfrak{A}$ *is a sequence of* i.d.'s $I_{\mathfrak{A}}(0), \cdots, I_{\mathfrak{A}}(k)$ *where* $k = n + \Sigma_i L(t_i)$ *such that*:

1) $I_{\mathfrak{A}}(0) = \langle s_0, mt_1\$, mt_2\$; \cdots, mt_n\$ \rangle$.
2) *For* $0 \geq j < k$, *if* $I_{\mathfrak{A}}(j) = \langle s, x_1 m \sigma_1 y_1, \cdots, x_n m \sigma_n y_n \rangle$ *for* σ_i *in* $\Sigma \cup \{\$\}$, *then, if* $f(s) = r$, $I_{\mathfrak{A}}(j+1) = \langle M(s, \sigma_r), w_1, \cdots, w_n \rangle$ *where* $w_r = x_r \sigma_r m y_r$ *and, for* $i \neq r$, $w_i = x_i m \sigma_i y_i$.
3) $I_{\mathfrak{A}}(k) = \langle s, w_1, \cdots, w_n \rangle$ *where s is in* $\{s_A, s_R\}$, *and* $w_i = t_i\$m$ *for* $i = 1, \cdots, n$.

An n-DA $\mathfrak{A}$ accepts an n-tuple $\langle t_1, \cdots, t_n \rangle$ iff there is a computation $I_{\mathfrak{A}}(0), \cdots, I_{\mathfrak{A}}(k)$ of $\langle t_1, \cdots, t_n \rangle$ such that the state of $\mathfrak{A}$ at time k is s_A. $\mathfrak{D}_n$ is the class of sets of n-tuples of tapes accepted by n-DA's.

Of special interest is the set of n-tuples of tapes

$$E_n = \{\langle t_1, t_2, \cdots, t_n \rangle : t_1 = t_2 = \cdots = t_n\}$$

in $\mathfrak{D}_n$ which defines the n-ary diagonal relation.

Theorem 1. *A set of tapes A is in* $\mathfrak{M}_n$ *if and only if there is a set of n-tuples of tapes B in* $\mathfrak{D}_n$ *such that A is the projection, on any coordinate, of* $B \cap E_n$. *That is,* $t \in A$ *iff* $\langle t, \cdots, t \rangle \in B$.

Proof: If A is defined by the n-FA $\mathfrak{A} = (K, M, h, s_0)$ then B is defined by the n-DA $\mathfrak{B} = (K, M, f, s_0)$ where for any state s in K, $f(s)$ in $\mathfrak{B}$ is equal to $h(s)$ in $\mathfrak{A}$. For the converse, we apply the inverse mapping. Thus, in essence, we interchange the tape- and head-selector functions to go from one class to the other. Q.E.D.

A context-free grammar (CFG) is a 4-tuple $\mathfrak{G} = (V, T, P, s)$ where V is a finite set, T is a subset of V, s is in $V - T$, and P is a subset of $(V - T) \times V^*$. For b in $V - T$ and y in V^*, we write $b \rightarrow y$ if (b, y) is in P; for words x and y over V, $x \Rightarrow y$ if $x = ubv$, $y = uzv$, and $b \rightarrow z$; $x \overset{*}{\Rightarrow} y$ if either $x = y$ or if there is a sequence $z_0, \cdots, z_r$ of words over V such that $x = z_0$, $y = z_r$ and $z_i \Rightarrow z_{i+1}$ for $i = 0, \cdots, r - 1$. The language generated by $\mathfrak{G}$, denoted $L(\mathfrak{G})$, is the set of strings over T, $L(\mathfrak{G}) = \{w : s \overset{*}{\Rightarrow} w\}$. A language L is context-free (is a CFL) if $L = L(\mathfrak{G})$ for some CFG $\mathfrak{G}$.

Theorem 1 now yields:

Theorem 2. *For* $n > 1$, *there is a set in* $\mathfrak{M}_n$ *which is not context-free.*

Proof: We show there is a non-CF set in $\mathfrak{M}_2$. Since, for all k, $\mathfrak{M}_k$ is obviously contained in $\mathfrak{M}_{k+1}$, the result will follow: Let $B = \{\langle 0^{m_1}1^{m_2}0^{m_3}, 0^{m_4}1^{m_1}0^{m_2} \rangle : m_i \geq 1\}$.

One can easily see that B is in $\mathfrak{D}_2$. The 2-DA moves the READ head on tape 2 past the initial string of 0's to the first 1. It then checks that the initial string of 0's on tape 1 is of the same length as the string of 1's on tape 2, and that the string of 1's on tape 1 is of the same length as the terminal string of 0's on tape 2. Now, by Theorem 1, there is a set A in $\mathfrak{M}_2$ which is the projection, on either coordinate, of $B \cap E_2$. Thus,

$$A = \{0^n1^n0^n : n \geq 1\}$$

which is not context-free (Bar-Hillel et al.[6]). Q.E.D.

We now determine the relationships among the $\mathfrak{M}_i$.

- *Relationship among classes* $\mathfrak{M}_i$

Theorem 3. *The* $\mathfrak{M}_i$ *form a hierarchy; that is, for all n,* $\mathfrak{M}_n$ *is a proper subset of* $\mathfrak{M}_{n+1}$.

Proof: We show that, for $n \geq 1$, the set

$$B_n = \{0^{m_1}10^{m_2}1 \cdots 0^{m_r}110^{m_r}1 \cdots 0^{m_2}10^{m_1} : m_i \geq 1\},$$

where $r = n(n+1)/2$ is in $\mathfrak{M}_{n+1} - \mathfrak{M}_n$.

a) We show that $\mathfrak{B}_n$ is in $\mathfrak{M}_{n+1}$ by describing the action of an $(n+1)$-FA $\mathfrak{A}$ which defines $\mathfrak{B}_n$. We intersperse examples of the processing by a 4-FA of a member of $\mathfrak{B}_3$; let this sample tape be

$$t = 0^a10^b10^c10^d10^e10^f110^f10^e10^d10^c10^b10^a$$

Since every member of $\mathfrak{B}_n$ has $n(n+1) = 2r$ blocks of 0's, we may unambiguously refer to the k^{th} block of 0's from the left as block k.

STEP 1(a): Dispatch head 1 to block $2r - n + 1$, and dispatch head k, for $k = 2, \cdots, n+1$ to block $k - 1$.

TAPE: $0^a10^b10^c10^d10^e10^f110^f10^e10^d10^c10^b10^a$

HEAD: $\uparrow$ 2 $\uparrow$ 3 $\uparrow$ 4 $\uparrow$ 1

STEP 1(b): Use (1) head 1 and head $n+1$ to check equality of blocks $2r - n + 1$ and n; (2) head 1 and head n

to check equality of blocks $2r - n + 2$ and $n - 1$; $\cdots$ (n) head 1 and head 2 to check equality of blocks $2r$ and 1.

TAPE†: $\mathit{0}^a\mathit{1}\mathit{0}^b 1 0^c 1 0^d 1 0^e 1 0^f 1 1 0^f 1 0^e 1 0^d 1 0^c 1 0^b 1 0^a$
HEAD: 2 3 4 1

STEP 2(a): Dispatch head 2 to block $2r - 2n + 2$, and dispatch head k, for $k = 3, \cdots, n + 1$ to block $n + k - 2$.

TAPE: $0^a 1 0^b 1 0^c 1 0^d 1 0^e 1 0^f 1 1 0^f 1 0^e 1 0^d 1 0^c 1 0^b 1 0^a$
HEAD: 3 4 2 1

STEP 2(b): Use: (1) head 2 and head $n + 1$ to check equality of blocks $2r - 2n + 2$ and $2n - 1$; (2) head 2 and head n to check equality of blocks $2r - 2n + 3$ and $2n - 2$; $\cdots$ ($n - 1$) head 2 and head 3 to check equality of blocks $2r - n$ and $n + 1$.

TAPE: $0^a 1 0^b 1 0^c 1 0^d 1 0^e 1 0^f 1 1 0^f 1 0^e 1 0^d 1 0^c 1 0^b 1 0^a$
HEAD: 3 4 2 1

STEP n — 1(a): Dispatch head $n - 1$ to block $r + 2$; and dispatch head n to block $r - 2$ and head $n + 1$ to block $r - 1$.

STEP n — 1(b): Use (1) head $n - 1$ and head $n + 1$ to check equality of blocks $r + 2$ and $r - 1$; (2) head $n - 1$ and head n to check equality of blocks $r + 3$ and $r - 2$.

STEP n(a): Dispatch head n to block $r + 1$, and dispatch head $n + 1$ to block r.

TAPE: $0^a 1 0^b 1 0^c 1 0^d 1 0^e 1 0^f 1 1 0^f 1 0^e 1 0^d 1 0^c 1 0^b 1 0^a$
HEAD: 4 3 2 1

STEP n(b): Use head n and head $n + 1$ to check for equality of blocks $r + 1$ and r.

TAPE: $0^a 1 0^b 1 0^c 1 0^d 1 0^e 1 0^f 1\ 1 0^f 1\ 0^e 1 0^d 1\ 0^c 1 0^b 1 0^a$
HEAD: 4 3 2 1

The interested reader can easily fill in the details to verify that $\mathfrak{A}$ does indeed define $\mathfrak{B}_n$. Thus $\mathfrak{B}_n$ is in $\mathfrak{M}_{n+1}$.

b) We now show that $\mathfrak{B}_n$ is not in $\mathfrak{M}_n$. The demonstration of this assertion is facilitated by the following observations.

Observation 1: The processing of any member of $\mathfrak{B}_n$ consists of comparing blocks $r - k + 1$ and $r + k$, where k ranges from 1 to r. This processing, then, involves $r = n(n + 1)/2$ comparisons.

Observation 2: The length of any block of 0's is not bounded. (This follows from the condition "$m_i \geq 1$" in the definition of $\mathfrak{B}_n$.)

Observation 3: By definition of our model, the READ heads of a multi-head FA are restricted to unilateral motion on the input tape.

Now assume that some n-FA $\mathfrak{A}$ defines $\mathfrak{B}_n$. If heads i and j of $\mathfrak{A}$ are used to compare blocks $r - k + 1$ and $r + k$ of a tape in $\mathfrak{B}_n$, for some k in $\{1, \cdots, r\}$, then we claim that this pair of heads cannot be used to compare any other pair of blocks on the tape. Let us assume that heads i and j are also used to compare blocks $r - k' + 1$ and $r + k'$, where $k' > k$ (a symmetric argument works if $k' < k$). By Observation 2, neither of these comparisons can be made by the finite-state memory alone. Thus, when the blocks $r - k + 1$ and $r + k$ are compared, one of the heads, say i, must be scanning block $r - k + 1$, while the other is scanning block $r + k$. A similar assertion holds for blocks $r - k' + 1$ and $r + k'$. One can easily see from the topology of the situation that the assumption that heads i and j are used for both comparisons must violate Observation 3. Thus each pair i and j of READ heads can be used for at most one comparison. It follows that $\mathfrak{A}$, having only n READ heads, can make at most $n(n - 1)/2$ comparisons (this being the number of pairs of heads of $\mathfrak{A}$). By Observation 1, $\mathfrak{A}$ cannot possibly define $\mathfrak{B}_n$. Thus $\mathfrak{B}_n$ is not in $\mathfrak{M}_n$. Q.E.D.

Since $\mathfrak{B}_n$ is clearly a CFL, we get

Corollary 3.1 *For all n, there is a* CFL *in* $\mathfrak{M}_{n+1} - \mathfrak{M}_n$.

We are now in a position to deduce the Boolean closure properties of $\mathfrak{M}_n$.

- *Closure properties*

Theorem 4. (a) *For all n,* $\mathfrak{M}_n$ *is closed under complementation.* (b) *for* $n > 1$, $\mathfrak{M}_n$ *is not closed under the operations of union or intersection.* (c) $\bigcup_r \mathfrak{M}_r$ *is a Boolean algebra.*

Proof: (a) If $\mathfrak{A} = (K, M, h, s_0)$ is an n-FA defining A, then the n-FA $\mathfrak{B} = (K, M', h, s_0)$ defines $\bar{A}$, where, for s in K:

$$M'(s, \sigma) = M(s, \sigma) \quad \text{for} \quad \sigma \quad \text{in} \quad \mathbf{\Sigma},$$

$$M'(s, \$) = \begin{cases} s_A & \text{if} \quad M(s, \$) = s_R \\ s_R & \text{if} \quad M(s, \$) = s_A \\ M(s, \$) & \text{otherwise.} \end{cases}$$

It follows from this construction that if A in $\mathfrak{M}_n$ is the projection, on any coordinate, of $B \cap E_n$ for B in $\mathfrak{D}_n$, then $\bar{A}$ is the projection on any coordinate of $\bar{B} \cap E_n$. Thus the mapping of Theorem 1 preserves complements.

(b) We show that, for $n > 1$, $\mathfrak{M}_n$ is not closed under intersection.

The following two sets are in $\mathfrak{M}_n$: ($\mathfrak{B}_{n-1}$ is as in Theorem 3)

$$A = (\{0\}^*\{1\})^n \mathfrak{B}_{n-1}(\{1\}\{0\}^*)^n$$

$$C = \{0^{m_1}1 \cdots 0^{m_n}10^{m_{n+1}}1 \cdots 0^{m_r}110^{m_{r+1}}1 \cdots 0^{m_{2r-n}} 10^{m_n}1 \cdots 10^{m_1}\colon m_i \geq 1 \quad \text{and} \quad r = n(n + 1)/2\}.$$

However, $A \cap C = \mathfrak{B}_n$ which, by Theorem 3 is in $\mathfrak{M}_{n+1} - \mathfrak{M}_n$. We conclude that $\mathfrak{M}_n$ is not closed under intersection.

† Those blocks of zero's which have been processed are in italics.

By part (a) and DeMorgan's law, we conclude that $\mathfrak{M}_n$ is not closed under union for $n > 1$.

(c) By part (a), $\mathfrak{M}_n$ is closed under complementation for all n.

It is obvious that, if A is in $\mathfrak{M}_n$, and B is in $\mathfrak{M}_m$, then both $A \cup B$ and $A \cap B$ are in $\mathfrak{M}_{m+n}$.

It follows that $\bigcup_r \mathfrak{M}_r$ is closed under the Boolean operations. Q.E.D.

Remark 2. For $n > 1$ the class of sets of tapes defined by nondeterministic n-FA's properly includes $\mathfrak{M}_n$.

Proof: Nondeterministic n-FA's are clearly closed under union.

In Theorem 2, we showed that there are sets in $\mathfrak{M}_n$, for all $n > 1$, that are not context-free. The following remark is, however, obvious.

Remark 3. For all n, $\mathfrak{M}_n$ is a subset of the class of context sensitive languages.†

In view of this remark, it is natural to wonder if the union over all $\mathfrak{M}_n$ exhausts the CSL's. If not, we wonder if the union exhausts at least the CFL's. We now exhibit a CFL which is in no $\mathfrak{M}_n$ for any n. Thus, the answer to both questions is in the negative.

Lemma 1. *If A is in $\mathfrak{M}_n$ and B is a regular set, then both $A \cup B$ and $A \cap B$ are in $\mathfrak{M}_n$.*

Proof: Since, for all n, $\mathfrak{M}_n$ is closed under complementation, it will suffice to show that $A \cap B$ is in $\mathfrak{M}_n$.

Let A be defined by the n-FA $(K, s_A, s_R, \mathbf{\Sigma}, \$, h, M, s_0)$, and let B be defined by the 1-FA $(L, t_A, t_R, \mathbf{\Sigma}, \$, N, t_0)$. (Note that the head selector function for a 1-FA, being degenerate, can be ignored). The set $A \cap B$ is then defined by the n-FA $(J, u_A, u_R, \mathbf{\Sigma}, \$, g, P, u_0)$ where

(1) $J = K \times (L \cup \{t_A\})$;

(2) $u_A = \langle s_A, t_A \rangle$; u_R is a special symbol, not in J;

(3) $\mathbf{\Sigma}$, \$ are as in Definition 1;

(4) for $\langle s, t\rangle$ in $K \times (L \cup \{t_A\})$, $g(\langle s, t\rangle) = h(s)$;

(5) for $\langle s, t\rangle$ in $K \times (L \cup \{t_A\})$,

(a) if σ is in $\mathbf{\Sigma}$,

$$P(\langle s, t\rangle, \sigma) = \begin{cases} \langle M(s, \sigma), N(t, \sigma)\rangle & \text{if } h(s) = 1, \\ \langle M(s, \sigma), t\rangle & \text{if } h(s) \neq 1; \end{cases}$$

$$\text{(b)}\quad P(\langle s, t\rangle, \$) = \begin{cases} u_R & \text{if } h(s) = 1 \text{ and } N(t, \$) = t_R, \text{ or if } m(s; \$) = s_R, \\ \langle M(s, \$), t_A\rangle & \text{if } h(s) = 1 \text{ and } N(t, \$) = t_A, \\ \langle M(s, \$), t\rangle & \text{if } h(s) \neq 1; \end{cases}$$

(6) $u_0 = \langle s_0, t_0 \rangle$.

In essence, the 1-FA defining B "shares" the first READ head of the n-FA defining A. A tape is accepted iff both automata would have accepted it; i.e. iff the tape is in $A \cap B$. Q.E.D.

Remark 4. The proof of Lemma 1 can be modified to show that, if A is in $\mathfrak{M}_n$ and B in $\mathfrak{M}_m$, then both $A \cup B$ and $A \cap B$ are in $\mathfrak{M}_{n+m-1}$.

Theorem 3 and Lemma 1 combine to yield

Theorem 5. *There is a CFL which is in $\mathfrak{M}_n$ for no integer n.*

Proof: Consider the set $A = \{t11t^T : t \text{ is in } \mathbf{\Sigma}^* - \mathbf{\Sigma}^*11\mathbf{\Sigma}^*\}$. (If t is a tape, then t^T is the *reversal* of t). Clearly A is a CFL.

Assume that, for some k, A is in $\mathfrak{M}_k$.

Consider the set $C = (00^*1)^r(100^*)^r$, where $r = k(k+1)/2$. Since C is a regular set, it follows from Lemma 1 that $A \cap C$ is in $\mathfrak{M}_k$. However, $A \cap C = \mathfrak{B}_k$ which by Theorem 3, is in $\mathfrak{M}_{k+1} - \mathfrak{M}_k$, a contradiction.

We conclude there is no integer n such that A is in $\mathfrak{M}_n$. Q.E.D.

Remark 5. It can be shown that A is a deterministic CFL. Thus, there is a deterministic CFL which is in $\mathfrak{M}_n$ for no integer n.

Theorem 3 and Lemma 1 further combine to yield the following theorem which determines the Kleene closure properties of $\mathfrak{M}_n$.

Theorem 6. *For $n > 1$, $\mathfrak{M}_n$ is not closed under the operations of concatenation, reversal, or closure.*

Proof: We consider first the operations of concatenation and reversal.

Let A and C be as in the proof of Theorem 4(b). As was shown there, $A \cap C$ is not in $\mathfrak{M}_n$; by Theorem 4(a), $\bar{A} \cup \bar{C}$ is not in $\mathfrak{M}_n$. However, since both $\bar{A}$ and $\bar{C}$ are in $\mathfrak{M}_n$, if c is a symbol not in the alphabet $\mathbf{\Sigma}$ over which A and C are encoded, then the set $\{c\}\bar{A}\{c\} \cup \bar{C}$ is in $\mathfrak{M}_n$. Let us denote this latter set by D.

Obviously the set $\{c, cc\}$, being finite, is in $\mathfrak{M}_1$. We claim that the set $\{c, cc\}D$ is not in $\mathfrak{M}_n$.

Assume, for contradiction, that the set is in $\mathfrak{M}_n$. It then follows, by Lemma 1, that the set $F = \{c, cc\}D \cap \{cc\} \cdot (\mathbf{\Sigma}^* \cup \mathbf{\Sigma}^*\{c\})$, which is the intersection of a set in $\mathfrak{M}_n$ and a regular set, is also in $\mathfrak{M}_n$. However, $F = \{cc\} \cdot (\bar{A}\{c\} \cup \bar{C})$ is not in $\mathfrak{M}_n$. Since $\bar{A}$ is in $\mathfrak{M}_n - \mathfrak{M}_{n-1}$, the n-FA cannot afford to send a head down to the end of the tape to determine whether or not there is a c preceding the \$. Thus $\bar{A}\{c\} \cup \bar{C}$ in $\mathfrak{M}_n$ would imply that $\bar{A} \cup \bar{C}$ is in $\mathfrak{M}_n$ which we know not to be the case.

We conclude that $\mathfrak{M}_n$ is not closed under concatenation.

Since A and C are symmetric sets (S is symmetric if $S = S^T$), $\bar{A}$ and $\bar{C}$, and thus D, are symmetric also. Thus, since $D\{c, cc\}$ is in $\mathfrak{M}_n$ while $(D \cdot \{c, cc\})^T = \{c, cc\} \cdot D^T = \{c, cc\} \cdot D$ is not in $\mathfrak{M}_n$, we conclude that $\mathfrak{M}_n$ is not closed under reversal.

To show that $\mathfrak{M}_n$ is not closed under closure, consider

† For definition of "context-sensitive" see Kuroda.[7]

the set $(\{c\}\bar{A} \cup \bar{C})\{c\} \cup \{c\}$ where A, C, and c are as above. Clearly this set, call it G, is in $\mathfrak{M}_n$. We claim that G^* is not in $\mathfrak{M}_n$.

Assume for contradiction that G^* is in $\mathfrak{M}_n$. It follows from Lemma 1 that the set $H = G^* \cap \{c\}\Sigma^*\{c\}$ is also in $\mathfrak{M}_n$. However, $H = \{c\}\cdot(\bar{A} \cup \bar{C})\cdot\{c\}$, which is clearly not in $\mathfrak{M}_n$ since $\bar{A} \cup \bar{C}$ is not.

We conclude that $\mathfrak{M}_n$ is not closed under closure for $n > 1$. Q.E.D.

Chapter II. Decision properties

We assume familiarity with the concepts of a Turing Machine (TM), the instantaneous description of a TM, the computation by a TM, and the concept of recursive undecidability. We employ the formalism of Davis.[8]

- *The halting problem*

To decide, given a TM $\mathfrak{T}$, whether or not the computation by $\mathfrak{T}$ with a completely blank initial tape is finite; that is, whether or not there exists an integer k such that the sequence $I_1, \cdots, I_k$ of instantaneous descriptions (i.d.'s) of $\mathfrak{T}$ is the computation by $\mathfrak{T}$ of a completely blank initial tape and I_k is a halting configuration.

We state without proof the following well-known result.

Theorem 7. *The halting problem for* TM's *is recursively unsolvable.*

We shall show several problems to be undecidable for n-FA's by showing that their decidability would imply the decidability of the halting problem. We shall say that a problem is undecidable for $\mathfrak{M}_n$ if it is undecidable for the class of n-FA's.

Lemma 2. *If a problem P is undecidable for $\mathfrak{M}_2$, then it is undecidable for $\mathfrak{M}_n$ for all $n \geq 2$.*

Proof: Any 2-FA can be considered as an n-FA in which only two of the READ heads are used.

We need, therefore, to prove undecidability results only for the class of 2-FA's; the generalization will follow by Lemma 2.

Theorem 8. *The following problems are recursively undecidable for sets A and B in $\mathfrak{M}_n$ for $n > 1$:*

(a) Is $A = \phi$?
(b) Is $A = \Sigma^*$?
(c) Is A finite?
(d) Is A regular?
(e) Is A context-free?
(f) Is A in $\mathfrak{M}_{n-1}$?
(g) Is $A \cap B = \phi$?
(h) Is $A \subseteq B$?
(i) Is $A = B$?

Proof: (a) This proof is a modification of one suggested by S. A. Greibach.[9] Assume, for contradiction, that, given an n-FA $\mathfrak{A}$, one could effectively decide whether or not $T(\mathfrak{A}) = \phi$.

Consider the class of sets defined thus: Let $\mathfrak{T}$ be a TM. Associate with $\mathfrak{T}$ the set of tapes:

$H(\mathfrak{T}) = \{I_{i_1}\beta I_{i_2}\beta \cdots \beta I_{i_k} : I_{i_1}, \cdots, I_{i_k}$ is a computation by $\mathfrak{T}$ when $\mathfrak{T}$ is started in its initial state on a completely blank tape; I_{i_k} is a halting configuration of $\mathfrak{T}$; and β is not in the alphabet over which the i.d.'s I_{i_j} are encoded$\}$.

We claim that, for all TM's $\mathfrak{T}$, $H(\mathfrak{T})$ is in $\mathfrak{M}_2$: The processing of a tape by the 2-FA proceeds in three stages.

(1) The 2-FA uses head 1 to check that I_{i_1} is a valid initial configuration of $\mathfrak{T}$; i.e., that $I_{i_1} = q_0B$ where q_0 is the initial state of $\mathfrak{T}$ and B is the blank symbol.

(2) The 2-FA uses both heads concurrently to check that $I_{i_{j+1}}$ is a valid consequent of I_{i_j} $(j = 1, \cdots, k-1)$. This is possible since if $I_{i_j} = xS'q_rSy$ for symbols S, S', state q_r and (possibly null) tapes x and y, then:

(*a*) if $q_rSS''q_s$ is an instruction of $\mathfrak{T}$,

$$I_{i_{j+1}} = xS'q_sS''y;$$

(*b*) if q_rSRq_s is an instruction of $\mathfrak{T}$,

$$I_{i_{j+1}} = xS'Sq_sy;$$

(*c*) if q_rSLq_s is an instruction of $\mathfrak{T}$,

$$I_{i_{j+1}} = xq_sS'Sy.$$

Thus, only three symbols of an i.d. need be examined.

(3) Finally, when head 1 encounters the end-tape symbol, head 2 is used to check that I_{i_k} is a halting configuration of $\mathfrak{T}$; i.e., that I_{i_k} is of the form $xS'qSy$, and no instruction of $\mathfrak{T}$ begins with qS. Since these three stages can all be performed by a 2-FA, it is clear that $H(\mathfrak{T})$ is in $\mathfrak{M}_2$.

Assuming the emptiness problem for $\mathfrak{M}_2$ to be solvable we could, in particular, given any TM $\mathfrak{T}$, decide whether or not $H(\mathfrak{T}) = \phi$. However, $H(\mathfrak{T}) = \phi$ iff the TM $\mathfrak{T}$ does not halt when started in its initial state on a completely blank tape. Thus, the solvability of the emptiness problem for $\mathfrak{M}_2$ implies the solvability of the Halting Problem, contradicting Theorem 7. The result follows.

Let us denote by C the class $\{H(\mathfrak{T}) : \mathfrak{T}$ is a TM$\}$.

(b) Assuming that the problem were solvable, we could, since $\mathfrak{M}_n$ is closed under complementation, solve the emptiness problem for C, contradicting part (a).

(c) Assume, for contradiction, that the finiteness problem were solvable for $\mathfrak{M}_n$. Let $H(\mathfrak{T})$ be as in part (a), and consider the set

$$Y(\mathfrak{T}) = H(\mathfrak{T})\{\gamma\}\{t\beta t : t \in \Sigma^* \quad \text{and} \quad \beta \notin \Sigma\},$$

where γ is not in the alphabet over which $H(\mathfrak{T})$ is encoded. $Y(\mathfrak{T})$ is in $\mathfrak{M}_2$.

If the finiteness problem were solvable, we could, in particular, decide whether or not, given a TM $\mathfrak{T}$, $Y(\mathfrak{T})$ were finite. However, $Y(\mathfrak{T})$ is finite iff $Y(\mathfrak{T}) = \phi$ iff $H(\mathfrak{T}) = \phi$. Thus the solvability of the finiteness problem implies the solvability of the emptiness problem for C contradicting part (a).

(d) and (e) Assume, for contradiction, that, given an n-FA $\mathfrak{A}$ one could effectively decide whether or not $T(\mathfrak{A})$ were regular (CF). We could then, in particular, decide whether or not the set $Y(\mathfrak{T})$ were regular (CF) for any TM $\mathfrak{T}$. However, since the set $\{t\beta t : t \in \Sigma^* \text{ and } \beta \notin \Sigma\}$ is not CF, $Y(\mathfrak{T})$ is regular (CF) iff $Y(\mathfrak{T}) = \phi$. Thus the solvability of these reduction problems implies the solvability of the emptiness problem for C, contradicting part (a). The result follows.

(f) We recall the definition of the set $\mathfrak{B}_n$ of Theorem 3. Consider, for a TM $\mathfrak{T}$, the set of tapes: $Z(\mathfrak{T}) = H(\mathfrak{T})\{\gamma\}\mathfrak{B}_{n-1}$ where γ is not in the alphabet over which $H(\mathfrak{T})$ is encoded.

Assume, for contradiction, that, given an n-FA $\mathfrak{A}$, one could effectively decide whether or not $T(\mathfrak{A})$ were in $\mathfrak{M}_{n-1}$. One could, in particular, decide whether or not $Z(\mathfrak{T})$ were in $\mathfrak{M}_{n-1}$ for any TM $\mathfrak{T}$. Since $H(\mathfrak{T})$ is in $\mathfrak{M}_2$, and since $\mathfrak{B}_{n-1}$ is in $\mathfrak{M}_n - \mathfrak{M}_{n-1}$, the set $Z(\mathfrak{T})$ is in $\mathfrak{M}_n - \mathfrak{M}_{n-1}$ unless $Z(\mathfrak{T}) = \phi$. However, $Z(\mathfrak{T}) = \phi$ iff $H(\mathfrak{T}) = \phi$.

Thus, the solvability of this reduction problem implies the solvability of the emptiness problem for C, contradicting part (a).

(g) Since, for all n, Σ^* is in $\mathfrak{M}_n$, if the disjointness problem for $\mathfrak{M}_n$ were solvable, we could decide, for any n-FA $\mathfrak{A}$, whether or not $T(\mathfrak{A}) \cap \Sigma^* = T(\mathfrak{A}) = \phi$, contradicting part (a).

(h) Assuming, for contradiction, that the containment problem were solvable, we could solve the disjointness problem for $\mathfrak{M}_n$, since $A \cap B = \phi$ iff $A \subseteq \bar{B}$. The result thus follows from part (g), and the effective closure of $\mathfrak{M}_n$ under complementation.

(i) Assuming, for contradiction, that the problem were solvable, we could, in particular, decide, given an n-FA $\mathfrak{A}$, whether or not $T(\mathfrak{A}) = \phi$, contradicting part (a). Q.E.D.

From Theorems 1 and 8, we obtain a result that strengthens the known results on the undecidability of the containment and emptiness of intersection problems for sets in $\mathfrak{D}_n$. We recall the set E_n of Theorem 1.

Theorem 9. *The following problems are recursively unsolvable for A in $\mathfrak{D}_n$:*

a) Is $A \cap E_n = \phi$?

b) Is $E_n \subseteq A$?

c) Is $A \subseteq \bar{E}_n$?

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