Cross-Products and Rotations
in 2- and 3-Dimensional Euclidean Spaces

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§0: Abstract and Table of Contents

Applications of cross-products to geometrical problems in Euclidean 3-Space lead to formulas that are easier to manipulate with associative matrix multiplications than with conventional non-associative and anti-commutative cross-products. This thesis is supported by derivations of neat formulas for rotations and for solutions to nearest-point problems. However, regardless of their notation, many neat formulas in textbooks can be rendered numerically deceptive by roundoff unless they are evaluated extra-precisely. Otherwise extra effort must be expended to compute results at least about as accurately as the data deserve.

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Starting in 1981, these notes were assembled from lecture notes for classes in Sophomore Multi-Variable Calculus and Linear Algebra, and in Junior Numerical Analysis. Students and diligent readers are expected to fill in some omitted but routine algebraic manipulations and proofs.

§1: √(–I) in Euclidean 2-Space (a Summary and Review)

The operator $-I$ reverses vectors. In two dimensions it has a skew-symmetric square root $J := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ determined uniquely but for its sign by the two equations $J^2 = -I$ and $J^T = -J$. This operator J rotates the plane through a quarter turn; whether clockwise or counter-clockwise depends upon the respectively left- or right-handed orientation of the coordinate system. More generally, $\exp(\theta\cdot J) := I\cdot\cos(\theta) + J\cdot\sin(\theta)$ turns the plane through an angle θ . To construct a vector of length $\|\mathbf{u}\| = \sqrt{(\mathbf{u}^T\mathbf{u})}$ perpendicular to any given vector $\mathbf{u}$ in the Euclidean plane, form $J\mathbf{u}$. For any 2-by-2 matrix $B := [\mathbf{u}\ \ \mathbf{v}]$ we find that $B^TJB = J\cdot\mathbf{v}^TJ\mathbf{u} = J\cdot\det(B)$, which implies $\mathrm{Adj}(B) = -JB^TJ$. (Recall $\mathrm{Adj}(B) := \det(B)\cdot B^{-1}$ when $\det(B) \neq 0$.) Our formulas use associative matrix multiplication for the scalar product $\mathbf{v}^T\cdot\mathbf{w} = \mathbf{w}^T\cdot\mathbf{v}$ instead of the non-associative dot product $\mathbf{v}\bullet\mathbf{w}$ for reasons that will become increasingly persuasive in the following pages.

Because J is unchanged by rotations of coordinates, it can produce ostensibly coordinate-free solutions for many geometric problems in the Euclidean plane. For instance, the equation of a line through $\mathbf{v}$ perpendicular to $\mathbf{w}$ is $\mathbf{w}^T(\mathbf{x}-\mathbf{v}) = 0$; the equation of a line through $\mathbf{v}$ parallel to $\mathbf{u}$ is $\mathbf{u}^TJ(\mathbf{x}-\mathbf{v}) = 0$; two lines whose equations are $\mathbf{u}^T\mathbf{x} = \mu$ and $\mathbf{v}^T\mathbf{x} = ß$ intersect at a point $\mathbf{z} := J\cdot(\mathbf{u}\cdot ß - \mathbf{v}\cdot\mu)/(\mathbf{v}^TJ\mathbf{u})$. However, not every orthogonal change of basis (coordinates) leaves J unchanged; a *Reflection* $W = W^{-1} = W^T \neq \pm I$ changes J to $W^{-1}JW = W^TJW = J\cdot\det(W) = -J$, which reminds us that reflection reverses orientation in the plane.

Do you see why such a W must be a reflection? Why it must have the form $W = I - 2\mathbf{w}\mathbf{w}^T/\mathbf{w}^T\mathbf{w}$ for a suitable vector $\mathbf{w}$? Why $\det(W) = -1$? Can you confirm every unobvious assertion in the summary above?)

In many ways, but not all, J is to the Euclidean plane what $\imath := \sqrt{(-1)}$ is to the complex plane. J operates upon vectors in the plane but is not a vector in that plane, whereas $\imath$ is simultaneously a multiplicative operator and a vector in the complex plane. The two planes are topologically different, though often confused: Roughly speaking, the complex plane has just one point at infinity best visualized by *Stereographically* projecting the complex plane upon a sphere, whereas the Euclidean plane has a circle (or at least a line) at infinity. We won't pursue this here.

Cross-products of vectors in Euclidean 2-Space appear in restrictions to 2-space of formulas derived originally for vectors in Euclidean 3-Space. Consequently the 2-space interpretation of " $\mathbf{u}\times\mathbf{v}$ " often reduces to a scalar $\mathbf{u}\times\mathbf{v} = \mathbf{v}^TJ\mathbf{u}$. Because cross-products are neither associative nor commutative, triple products like " $\mathbf{u}\bullet\mathbf{v}\times\mathbf{w}$ ", " $\mathbf{u}\times\mathbf{v}\bullet\mathbf{w}$ " and " $\mathbf{u}\times\mathbf{v}\times\mathbf{w}$ " can generate confusion if parsed improperly. When all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ lie in the same Euclidean 2-Space, …

$\mathbf{u}\bullet(\mathbf{v}\times\mathbf{w})$ and $(\mathbf{u}\times\mathbf{v})\bullet\mathbf{w}$ should both be zero if they appear at all, and

$(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = -\mathbf{w}\times(\mathbf{u}\times\mathbf{v}) = J\mathbf{w}\cdot(\mathbf{v}^TJ\mathbf{u})$ in 2-space.

Whence come these formulas? From §§4-5. They will make sense after we have found and adopted a matrix notation for cross-products, which motivates the notes that follow.

§2: Cross-Products and Rotations in Euclidean 3-Space

Henceforth bold-faced lower-case letters $\mathbf{p}, \mathbf{q}, \mathbf{r}, \ldots, \mathbf{x}, \mathbf{y}, \mathbf{z}$ stand for real 3-dimensional column-vectors. Then row vector $\mathbf{p}^T := [p_1\ \ p_2\ \ p_3]$ is the transpose of column vector $\mathbf{p}$, and $\mathbf{p}^T\cdot\mathbf{q}$ is the scalar product $\mathbf{p}\bullet\mathbf{q}$ of row $\mathbf{p}^T$ and column $\mathbf{q}$. Euclidean length $\|\mathbf{p}\| := \sqrt{(\mathbf{p}^T\cdot\mathbf{p})}$.

Do not confuse the scalar $\mathbf{p}^T\cdot\mathbf{q} = \mathbf{q}^T\cdot\mathbf{p}$ with the 3–by–3 matrices (“dyads”) $\mathbf{p}\cdot\mathbf{q}^T \ne \mathbf{q}\cdot\mathbf{p}^T$ of rank 1, nor with the vector cross-product $\mathbf{p}\times\mathbf{q} = -\mathbf{q}\times\mathbf{p}$. Cross-products are important enough to justify introducing a notation “ $\mathbf{p}^{¢}$ ”, pronounced “ pee-cross”, for a 3-by-3 skew-symmetric ($\mathbf{p}^{¢T} = -\mathbf{p}^{¢}$) matrix of rank 2 defined by the vector cross-product thus: $\mathbf{p}\times\mathbf{q} = \mathbf{p}^{¢}\cdot\mathbf{q}$. Explicitly the matrix $\mathbf{p}^{¢} := \begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix}$. We shall see whence this comes after we see why we like it.

We prefer matrix notation for these geometrical entities because matrix multiplication is *associative*:

$$\mathbf{p}^T\cdot\mathbf{q}^{¢}\cdot\mathbf{r} = (\mathbf{p}^T\cdot\mathbf{q}^{¢})\cdot\mathbf{r} = \mathbf{p}^T\cdot(\mathbf{q}^{¢}\cdot\mathbf{r}) = \mathbf{p}\bullet(\mathbf{q}\times\mathbf{r}) \quad \text{and} \quad \mathbf{p}^{¢}\cdot\mathbf{q}^{¢}\cdot\mathbf{r} = (\mathbf{p}^{¢}\cdot\mathbf{q}^{¢})\cdot\mathbf{r} = \mathbf{p}^{¢}\cdot(\mathbf{q}^{¢}\cdot\mathbf{r}) = \mathbf{p}\times(\mathbf{q}\times\mathbf{r})$$

unlike scalar and cross-products; $(\mathbf{p}\bullet\mathbf{q})\cdot\mathbf{r} \ne \mathbf{p}\cdot(\mathbf{q}\bullet\mathbf{r})$ and $(\mathbf{p}\times\mathbf{q})\times\mathbf{r} \ne \mathbf{p}\times(\mathbf{q}\times\mathbf{r})$. Besides legibility, a matrix notation promotes simpler expressions, shorter proofs, and easier operator overloading in programming languages.

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§3: For Readers Reluctant to Abandon • and x Products

(Other readers can skip to the next string of asterisks.)

We’re not abandoning familiar locutions; we’re just writing most of them shorter. Compare the following classical formulas with their matrix equivalents for succinctness and ease of proof:

Triple Cross-Product: $(\mathbf{p}\times\mathbf{q})\times\mathbf{r} = \mathbf{q}\cdot\mathbf{p}\bullet\mathrm{r} - \mathbf{p}\cdot\mathbf{q}\bullet\mathrm{r}$ vs. $(\mathbf{p}^{¢}\cdot\mathbf{q})^{¢} = \mathbf{q}\cdot\mathbf{p}^T - \mathbf{p}\cdot\mathbf{q}^T$

Jacobi’s Identity: $\mathbf{p}\times(\mathbf{q}\times\mathbf{r}) + \mathbf{q}\times(\mathbf{r}\times\mathbf{p}) = -\mathbf{r}\times(\mathbf{p}\times\mathbf{q})$ vs. $\mathbf{p}^{¢}\cdot\mathbf{q}^{¢} - \mathbf{q}^{¢}\cdot\mathbf{p}^{¢} = (\mathbf{p}^{¢}\cdot\mathbf{q})^{¢}$

Lagrange’s Identity: $(\mathbf{t}\times\mathbf{u})\bullet(\mathbf{v}\times\mathbf{w}) = \mathbf{t}\bullet\mathbf{v}\cdot\mathbf{u}\bullet\mathbf{w} - \mathbf{u}\bullet\mathbf{v}\cdot\mathbf{t}\bullet\mathbf{w}$ vs. $(\mathbf{t}^{¢}\cdot\mathbf{u})^T\cdot(\mathbf{v}^{¢}\cdot\mathbf{w}) = \det([\mathbf{t}\ \ \mathbf{u}]^T\cdot[\mathbf{v}\ \ \mathbf{w}])$

Some things don’t change much; $\mathbf{p}\times\mathbf{q} = -\mathbf{q}\times\mathbf{p}$ becomes $\mathbf{p}^{¢}\cdot\mathbf{q} = -\mathbf{q}^{¢}\cdot\mathbf{p}$, so $\mathbf{p}^{¢}\cdot\mathbf{p} = \mathbf{o}$ (the zero vector), and $\mathbf{p}\bullet(\mathbf{q}\times\mathbf{r}) = \mathbf{p}^T\cdot\mathbf{q}^{¢}\cdot\mathbf{r} = \det([\mathbf{p}\ \ \mathbf{q}\ \ \mathbf{r}])$.

The notations’ difference becomes more pronounced as problems become more complicated. For instance, given a unit vector $\hat{\mathbf{u}}$ (with $\|\hat{\mathbf{u}}\| = 1$) and a scalar ψ, what orthogonal matrix $\mathrm{R} = (\mathrm{R}^T)^{-1}$ rotates Euclidean 3–space through an angle ψ radians around the axis $\hat{\mathbf{u}}$? In other words, $\mathrm{R}\cdot\mathbf{x}$ is to transform every vector $\mathbf{x}$ by rotating it through an angle ψ about an axis $\hat{\mathbf{u}}$ fixed through the origin $\mathbf{o}$.

An ostensibly simple formula $\mathrm{R} := \exp(\psi\cdot\hat{\mathbf{u}}^{¢})$ uses the skew-symmetric cross-product matrix $\hat{\mathbf{u}}^{¢}$ defined above. Here exp(…) is *not* the *array* exponential that is applied elementwise, but is the *matrix* exponential; think of $\mathrm{R} = \mathrm{R}(\psi)$ as a matrix-valued function of ψ that solves the differential equation $d\mathrm{R}/d\psi = \hat{\mathbf{u}}^{¢}\cdot\mathrm{R} = \mathrm{R}\,\hat{\mathbf{u}}^{¢}$ starting from the identity matrix $\mathrm{R}(0) = \mathrm{I}$. Given $\hat{\mathbf{u}}$ and ψ, an explicit formula for this R is $\mathrm{R} = \mathrm{I} + 2\cdot\big(\mathrm{I}\cdot\cos(\psi/2) + \hat{\mathbf{u}}^{¢}\cdot\sin(\psi/2)\big)\cdot\hat{\mathbf{u}}^{¢}\cdot\sin(\psi/2)$. Rewriting this expression with solely • and × products doesn’t improve it. Try that!

In what follows the formulas above will be first derived and then applied to a few examples.

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§4: A Derivation of Cross-Products in Euclidean 3-Space

What operators in Euclidean 3-space are analogous to the quarter-turn J in 2-space? Every rotation of 3-space is characterized by its *axis*, a line left unchanged by the rotation, and its angle of rotation about that axis. Let $\mathbf{v}$ be a nonzero vector parallel to such an axis. Analogous to $-\mathrm{I}$ in 2-space is the operator $\mathbf{v}\mathbf{v}^T/\mathbf{v}^T\mathbf{v} - \mathrm{I}$, which projects arbitrary vectors into the plane through $\mathbf{o}$ perpendicular to $\mathbf{v}$ and then reverses the projection through $\mathbf{o}$. That operator's skew-symmetric square root, determined (as we shall see) uniquely but for its sign by $\mathbf{v}$, is the analog of J, but different for every different direction $\mathbf{v}$. Consequently that square root is a discontinuous function of $\mathbf{v}$ at $\mathbf{v} = \mathbf{o}$. Multiplying that square root by $\|\mathbf{v}\|$ renders it continuous.

Hence we define the operator $\mathbf{v}^{¢}$ to be one of the solutions $\mathbf{v}^{¢} := \pm S$ of the equations

$$S^2 = \mathbf{v}\mathbf{v}^T - \mathbf{v}^T\mathbf{v}\cdot \mathrm{I} \quad \text{and} \quad S = -S^T .$$

To see why these equations determine S uniquely but for sign, choose an orthonormal basis with $\mathbf{v}/\|\mathbf{v}\|$ as its first basis vector and find a matrix representing S in that coordinate system. Every such matrix S must satisfy $S\mathbf{v} = \mathbf{o}$; here is why: Evidently $S^2\mathbf{v} = \mathbf{o}$, so $\det(S)^2 = \det(S^2) = 0$, and so $S\mathbf{z} = \mathbf{o}$ for some $\mathbf{z} \neq \mathbf{o}$; but then $S^2\mathbf{z} = \mathbf{o}$, and this implies that $\mathbf{z}$ is a scalar multiple of $\mathbf{v}$, whence follows $S\mathbf{v} = \mathbf{o}$ as claimed. Consequently, in the foregoing orthonormal coordinate system, every skew-symmetric solution S is represented by a matrix whose first row and column contain only zeros, whereupon the remaining 2-by-2 principal submatrix must be $\pm J\cdot\|\mathbf{v}\|$ as explained in the second sentence of §1. Thus, S is determined uniquely but for sign.

Given $\mathbf{v} = \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix}$, consider the solution $S := \begin{bmatrix} 0 & -\zeta & \eta \\ \zeta & 0 & -\xi \\ -\eta & \xi & 0 \end{bmatrix}$ of the equations $S\mathbf{v} = \mathbf{o}$ and $S = -S^T$;

this $S^2 = \mathbf{v}\mathbf{v}^T - \mathbf{v}^T\mathbf{v}\cdot \mathrm{I}$ too, which combines with the previous paragraph to imply $\mathbf{v}^{¢} = \pm S$. Its sign could be chosen arbitrarily but we set $\mathbf{v}^{¢} := +S$, thereby classifying the coordinate system as "right-handed". Note now that $\mathbf{v}^{¢}$ is a continuous function of $\mathbf{v}$. In summary, …

> For every vector $\mathbf{v}$ in Euclidean 3-space, the linear operator $\mathbf{v}^{¢}$ is a continuous linear function of $\mathbf{v}$ determined but for sign by the equations $(\mathbf{v}^{¢})^2 = \mathbf{v}\mathbf{v}^T - \mathbf{v}^T\mathbf{v}\cdot \mathrm{I}$ and $(\mathbf{v}^{¢})^T = -\mathbf{v}^{¢}$. Its sign is determined for every $\mathbf{v}$ by its sign for any one $\mathbf{v} \neq \mathbf{o}$ and by continuity.

The notation for $\mathbf{v}^{¢}$, pronounced "vee-cross", is inspired by the relation $\mathbf{v}^{¢}\mathbf{w} = \mathbf{v}\times\mathbf{w}$, where the latter cross-product coincides with the one defined in texts on vector analysis. Here are four of its properties:

$\mathbf{v}^{¢}\mathbf{w} = \mathbf{o}$ just when $\pm\mathbf{v}$ and $\mathbf{w}$ are parallel; this was proved using $\mathbf{z}$ above.

$\mathbf{v}^{¢}\mathbf{w} \perp \mathbf{v}$ because $\mathbf{v}^T\mathbf{v}^{¢}\mathbf{w} = -(\mathbf{v}^{¢}\mathbf{v})^T\mathbf{w} = \mathbf{o}^T\mathbf{w} = 0$.

$\mathbf{v}^{¢}\mathbf{w} \perp \mathbf{w}$ because $\mathbf{w}^T\mathbf{v}^{¢}\mathbf{w} = (\mathbf{w}^T\mathbf{v}^{¢}\mathbf{w})^T = \mathbf{w}^T(\mathbf{v}^{¢})^T\mathbf{w} = -\mathbf{w}^T\mathbf{v}^{¢}\mathbf{w} = 0$.

$\|\mathbf{v}^{¢}\mathbf{w}\|^2 = \|\mathbf{v}\|^2\,\|\mathbf{w}\|^2 - (\mathbf{v}^T\mathbf{w})^2$ because it is $-\mathbf{w}^T(\mathbf{v}^{¢})^2\mathbf{w}$, *etc.*

Combining the formula $\mathbf{v}^T\mathbf{w} = \|\mathbf{v}\|\cdot\|\mathbf{w}\|\cdot\cos\angle(\mathbf{v}, \mathbf{w})$ with the last equation proves that

$$\|\mathbf{v}^{¢}\mathbf{w}\| = \pm\|\mathbf{v}\|\cdot\|\mathbf{w}\|\cdot\sin\angle(\mathbf{v}, \mathbf{w})$$

with a sign that depends upon the orientation, if any, assigned to the angle $\angle(\mathbf{v}, \mathbf{w})$ when it has to be distinguished from $\angle(\mathbf{v}, \mathbf{w}) = -\angle(\mathbf{w}, \mathbf{v})$. Anyway,

$$\|\mathbf{v}^{¢}\mathbf{w}\| = \big|\text{ the area of a parallelogram with adjacent sides } \mathbf{v} \text{ and } \mathbf{w} \big| \,.$$

From the foregoing properties we infer by symmetry that $\mathbf{w}^{¢}\mathbf{v}$ must be one of $\pm\mathbf{v}^{¢}\mathbf{w}$ whenever they are nonzero because they are vectors with the same length and perpendicular to the same two nonparallel vectors $\mathbf{v}$ and $\mathbf{w}$. Trials with basis vectors for $\mathbf{v}$ and $\mathbf{w}$ show that

$$\mathbf{w}^{¢}\mathbf{v} = -\,\mathbf{v}^{¢}\mathbf{w} \,,$$

and this equation must persist for all $\mathbf{v}$ and $\mathbf{w}$ since both sides are continuous bilinear functions. This *anti-commutative* identity is a good reason to prefer the notation $\mathbf{v}^{¢}\mathbf{w}$ over $\mathbf{v}\times\mathbf{w}$; and later the preference will intensify when we find the triple cross-product *non-associative*. Besides, we shall need $\mathbf{v}^{¢}$ in isolation later to describe rotations.

§5: Triple Products

The scalar expression $\mathbf{u}^T\mathbf{v}^{¢}\mathbf{w}$ is linear in each vector separately, and reverses sign when any two vectors are swapped; this follows from anti-commutativity when $\mathbf{v}$ and $\mathbf{w}$ are swapped, from skew-symmetry of $\mathbf{v}^{¢}$ when $\mathbf{u}$ and $\mathbf{w}$ are swapped; and when $\mathbf{u}$ and $\mathbf{v}$ are swapped it follows from $\mathbf{u}^T\mathbf{v}^{¢}\mathbf{w} = -\mathbf{w}^T\mathbf{v}^{¢}\mathbf{u} = \mathbf{w}^T\mathbf{u}^{¢}\mathbf{v} = -\mathbf{v}^T\mathbf{u}^{¢}\mathbf{w}$. Compare this with the characterization of the determinant $\det([\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w}])$ as a functional, linear in each column of $[\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w}]$ separately, that reverses sign when any two columns are swapped. It follows that $\mathbf{u}^T\mathbf{v}^{¢}\mathbf{w}/\det([\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w}])$ must be a constant provided the denominator does not vanish. Setting matrix $[\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w}] = \mathrm{I}$ determines that constant to be 1 , whereupon continuity implies an important identity for all $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$:

$$\mathbf{u}^T\mathbf{v}^{¢}\mathbf{w} = \det([\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w}]) \,.$$

This formula can be confirmed by direct but tedious algebraic manipulation, and also by the following geometric argument:

Let parallelogram $\boldsymbol{P}$ have adjacent sides $\mathbf{v}$ and $\mathbf{w}$ so that its area $|\boldsymbol{P}| = \|\mathbf{v}^{¢}\mathbf{w}\| \ne 0$. Next let $\boldsymbol{Q}$ be a parallelepiped whose sides emanating from a vertex are $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$; then its volume is $|\boldsymbol{Q}| = \det([\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w}])$ and also

$$\begin{aligned} |\boldsymbol{Q}| &= |\boldsymbol{P}|\cdot\|\text{ projection of } \mathbf{u} \text{ onto the unit-normal to } \boldsymbol{P}\,\| \\ &= \|\mathbf{v}^{¢}\mathbf{w}\|\cdot\|\text{ projection of } \mathbf{u} \text{ onto } \mathbf{v}^{¢}\mathbf{w}/\|\mathbf{v}^{¢}\mathbf{w}\|\,\| = |\mathbf{u}^T\mathbf{v}^{¢}\mathbf{w}| \,. \end{aligned}$$

Now to confirm that $\mathbf{u}^T\mathbf{v}^{¢}\mathbf{w} = +\det([\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w}])$ try any three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, say the basis vectors, and then invoke continuity to cope with the case when $\mathbf{v}$ and $\mathbf{w}$ are (anti)parallel.

Almost as important as that determinantal formula is the *triple cross-product formula*

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = \mathbf{u}^{¢}\mathbf{v}^{¢}\mathbf{w} = \mathbf{v}\cdot\mathbf{u}^T\mathbf{w} - \mathbf{w}\cdot\mathbf{u}^T\mathbf{v} = (\mathbf{v}\mathbf{w}^T - \mathbf{w}\mathbf{v}^T)\mathbf{u} \,.$$

To prove this, note that it must be perpendicular to a vector $\mathbf{v}^{¢}\mathbf{w}$ perpendicular to both $\mathbf{v}$ and $\mathbf{w}$, and hence must lie in the plane of $\mathbf{v}$ and $\mathbf{w}$. Therefore $\mathbf{u}^{¢}\mathbf{v}^{¢}\mathbf{w} = \mathbf{v}\cdot ß - \mathbf{w}\cdot µ$ for some scalars ß and µ . Premultiplication by $\mathbf{u}^T$ reveals that $0 = \mathbf{u}^T\mathbf{v}\cdot ß - \mathbf{u}^T\mathbf{w}\cdot µ$ and therefore some scalar functional $f = \mathbf{u}^T\mathbf{v}/µ = \mathbf{u}^T\mathbf{w}/ß$ exists satisfying $\mathbf{u}^{¢}\mathbf{v}^{¢}\mathbf{w} = (\,\mathbf{v}\cdot\mathbf{u}^T\mathbf{w} - \mathbf{w}\cdot\mathbf{u}^T\mathbf{v}\,)/f$. Since both sides of this equation are linear in each of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ separately, f can vary with none of them; it must be a constant. Its value $f = 1$ can be found by substituting one basis vector for $\mathbf{u}$ and $\mathbf{w}$

and a second basis vector for $\mathbf{v}$. Alternatively, brute-force manipulation by a computerized algebra system like *Derive*, *Maple*, *Mathematica* or *Macsyma* can be used to confirm the triple cross-product formula. It is easier to remember in §3's form: $(\mathbf{v}^{¢}\mathbf{w})^{¢} = \mathbf{w}\mathbf{v}^T - \mathbf{v}\mathbf{w}^T$.

That formula shows that $(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = (\mathbf{u}^{¢}\mathbf{v})^{¢}\mathbf{w} = -\mathbf{w}^{¢}\mathbf{u}^{¢}\mathbf{v} = (\mathbf{v}\mathbf{u}^T - \mathbf{u}\mathbf{v}^T)\mathbf{w} \neq \mathbf{u}\times(\mathbf{v}\times\mathbf{w})$; the cross-product is *not* associative, though matrix multiplication *is* associative: $(\mathbf{u}^{¢}\mathbf{v}^{¢})\mathbf{w} = \mathbf{u}^{¢}(\mathbf{v}^{¢}\mathbf{w})$. That formula also confirms *Jacobi's Identity*:

$$\mathbf{u}^{¢}\mathbf{v}^{¢}\mathbf{w} + \mathbf{v}^{¢}\mathbf{w}^{¢}\mathbf{u} + \mathbf{w}^{¢}\mathbf{u}^{¢}\mathbf{v} = \mathbf{o}\ , \quad \text{or} \quad (\mathbf{u}^{¢}\mathbf{v})^{¢} = \mathbf{u}^{¢}\mathbf{v}^{¢} - \mathbf{v}^{¢}\mathbf{u}^{¢}\ ,$$

and helps to confirm *Lagrange's Identity*:

$$(\mathbf{t}^{¢}\cdot\mathbf{u})^T(\mathbf{v}^{¢}\cdot\mathbf{w}) \ = \ \mathbf{t}^T\mathbf{v}\cdot\mathbf{u}^T\mathbf{w} - \mathbf{u}^T\mathbf{v}\cdot\mathbf{t}^T\mathbf{w} \ = \det([\mathbf{t}\ \ \mathbf{u}]^T[\mathbf{v}\ \ \mathbf{w}])\ .$$

Since they are not long, you should work out the confirmations of these identities, which figure in both classical and Quantum mechanics.

§6: Rotations R about an Axis v in Euclidean 3-Space

If skew-symmetric matrix $S = -S^T$ is constant, the unique solution of the initial-value problem

$$R(0) = I \quad \text{and} \quad dR/d\tau = SR \ \text{ for all } t$$

is a matrix $R(\tau)$ that must be orthogonal; $R^T = R^{-1}$ because $d(R^TR)/d\tau = R^TS^TR + R^TSR = O$ and therefore $R^TR = I$ for all τ . This implies $\det(R)^2 = 1$ and then $\det(R) = +1$ because it is continuous for all τ . Thus, $R(\tau)$ is a proper rotation;– no reflection. It has a power series too:

$$R(\tau) = \exp(\tau\cdot S) = \textstyle\sum_{k\geq 0} \tau^k\cdot S^k/k!\ .$$

Now, every 3-by-3 skew-symmetric matrix S determines a vector $\mathbf{v}$ such that $S = \mathbf{v}^{¢}$; then $S\mathbf{v} = \mathbf{o}$, $S^2 = \mathbf{v}\mathbf{v}^T - \|\mathbf{v}\|^2\cdot I$, $S^3 = -\|\mathbf{v}\|^2\cdot S$, $S^4 = -\|\mathbf{v}\|^2\cdot S^2$, …, $S^{m+2k} = (-\|\mathbf{v}\|^2)^k\cdot S^m$ for $m > 0$. By taking odd and even terms separately in the series for $\exp(\tau\cdot\mathbf{v}^{¢})$ we condense it to

$$\begin{aligned} R(\tau) = \exp(\tau\cdot\mathbf{v}^{¢}) \ &= \ I + \big(1 - \cos(\tau\cdot\|\mathbf{v}\|)\big)\cdot(\mathbf{v}^{¢}/\|\mathbf{v}\|)^2 + \sin(\tau\cdot\|\mathbf{v}\|)\cdot\mathbf{v}^{¢}/\|\mathbf{v}\| \\ &= \ I + 2\sin(\tau\cdot\|\mathbf{v}\|/2)\cdot\big(\sin(\tau\cdot\|\mathbf{v}\|/2)\cdot\mathbf{v}^{¢}/\|\mathbf{v}\| + \cos(\tau\cdot\|\mathbf{v}\|/2)\cdot I\big)\cdot\mathbf{v}^{¢}/\|\mathbf{v}\|\ , \end{aligned}$$

thus providing a relatively simple and verifiable formula for the operator that rotates Euclidean 3-space through an angle $\tau\cdot\|\mathbf{v}\|$ about a given axis $\mathbf{v} \neq \mathbf{o}$. Its τ-derivative is extremely simple:

$$d\exp(\tau\cdot\mathbf{v}^{¢})/d\tau = \mathbf{v}^{¢}\cdot\exp(\tau\cdot\mathbf{v}^{¢}) = \exp(\tau\cdot\mathbf{v}^{¢})\cdot\mathbf{v}^{¢}\ .$$

(The $\mathbf{v}$-derivative would require a long expression too complicated to serve the didactic purposes of these notes.)

The converse problem is this: Given an orthogonal matrix $R = R^{T-1}$ that effects a proper rotation because $\det(R) = +1$, how can its axis $\mathbf{v}$ be determined? What seems the simplest way at first is to compute $R - R^T = 2\cdot\sin(\tau\cdot\|\mathbf{v}\|)\cdot\mathbf{v}^{¢}/\|\mathbf{v}\|$, which works provided $\sin(\tau\cdot\|\mathbf{v}\|)$ is not so tiny that roundoff in R obscures the desired result. Generally a more reliable procedure is to apply Gaussian elimination to solve the equation $(R-I)\mathbf{v} = o$ for a $\mathbf{v} \neq o$, or alternatively to compute an appropriate (not too small) column of $\mathrm{Adj}(R-I) = (\text{some scalar})\cdot\mathbf{v}\mathbf{v}^T$. This procedure works because $R-I$ is singular (to within roundoff) with rank 2 ; here is how we know this to be so:

Consider any eigenvalue μ of R ; this μ may be complex, in which case its eigenvector $\mathbf{z}$ is complex too, and we shall write $\mathbf{z}^*$ for its complex-conjugate transpose. Next we find that $|\mu|^2\cdot\mathbf{z}^*\mathbf{z} = (R\mathbf{z})^*(R\mathbf{z}) = \mathbf{z}^*R^TR\mathbf{z} = \mathbf{z}^*\mathbf{z} > 0$, whereupon $|\mu| = 1$. Now, R has three eigenvalues μ , the roots of the characteristic equation $\det(\mu I - R) = 0$. Because its coefficients are real, any of the three eigenvalues that are not real must come in complex-conjugate pairs whose product, their squared magnitude, must equal 1 . The product of all three eigenvalues is $\det(R) = +1$ too. Two cases can arise:

- If R has a non-real eigenvalue μ then μ^* is another and the third is $1/(\mu^*\mu) = 1$.
- Otherwise all three eigenvalues are real, namely ± 1 , and then $+1$ appears among them an odd number of times because their product is $+1$ too.

Thus $R–I$ must be singular; the axis $\mathbf{v}$ is an eigenvector of R belonging to eigenvalue $+1$.

Problem (hard): Show that $\mathrm{Adj}(R–I) = (3 – \mathrm{Trace}(R))\cdot\mathbf{v}\mathbf{v}^T/\|\mathbf{v}\|^2$ provided proper orthogonal $R \neq I$.

Must each proper orthogonal $R = \exp(\tau\cdot\hat{\mathbf{u}}^{¢})$ for some real τ and unit vector $\hat{\mathbf{u}}$ (with $\|\hat{\mathbf{u}}\| = 1$) ? Yes, and $\hat{\mathbf{u}} = \mathbf{v}/\|\mathbf{v}\|$ where $\mathbf{v}$ is the axis found above. To see why, change to a new orthonormal coordinate system with $\hat{\mathbf{u}}$ as its first basis vector. The matrix representing R in this new basis has $[1\ \ 0\ \ 0]$ and its transpose as first row and column. (Why?) The last principal 2-by-2 submatrix must be $\exp(\tau\cdot J)$ because it is proper orthogonal too; thus τ is determined. After changing back to the original basis we find $R = \exp(\pm\tau\cdot\hat{\mathbf{u}}^{¢})$. (We'll explain the $\pm$ sign later.)

§7: Constructing Rotations out of Reflections in a Euclidean Space of any Dimension ≥ 2

For any $\mathbf{w} \neq \mathbf{o}$ in *any* Euclidean space, $W := I – 2\mathbf{w}\mathbf{w}^T/\|\mathbf{w}\|^2$ is an orthogonal reflection.

Problem: Verify that $W = W^T = W^{–1}$, that $\mathbf{w} = –W\mathbf{w}$ is reversed by the reflection, and that it preserves the (hyper)plane of vectors $\mathbf{x}$ orthogonal to $\mathbf{w}$. Thus the reflection's mirror-plane satisfies the equation $\mathbf{w}^T\mathbf{x} = 0$. Verify too that $\det(W) = –1$ by applying the formula $\det(I – \mathbf{u}\mathbf{v}^T) = 1 – \mathbf{v}^T\mathbf{u}$. Can you prove this last formula?

Suppose distinct nonzero vectors $\mathbf{x}, \mathbf{y}, \mathbf{s}$ and $\mathbf{t}$ are given with $\|\mathbf{x}\| = \|\mathbf{y}\|$ and $\|\mathbf{s}\| = \|\mathbf{t}\|$ and $\mathbf{s}^T\mathbf{x} = \mathbf{t}^T\mathbf{y}$. (This last equation says that $|\angle(\mathbf{s}, \mathbf{x})| = |\angle(\mathbf{t}, \mathbf{y})|$.) We wish to construct a proper orthogonal R that rotates $\mathbf{x}$ to $R\mathbf{x} = \mathbf{y}$ and $\mathbf{s}$ to $R\mathbf{s} = \mathbf{t}$. We shall construct this $R := HW$ as a product of two orthogonal reflections: $W := I – 2\mathbf{w}\mathbf{w}^T/\|\mathbf{w}\|^2$ and $H := I – 2\mathbf{h}\mathbf{h}^T/\|\mathbf{h}\|^2$ in which $\mathbf{w} := \mathbf{x}–\mathbf{y}$ and $\mathbf{h} := W\mathbf{s}–\mathbf{t}$, except that if $W\mathbf{s} = \mathbf{t}$ then $\mathbf{h}$ may be any nonzero vector orthogonal to both $\mathbf{y}$ and $\mathbf{t}$ provided such a vector exists. (R might not exist in 2-space; why not?)

Problem: Verify that W swaps $\mathbf{x}$ and $\mathbf{y}$, and that H swaps $W\mathbf{s}$ and $\mathbf{t}$ while preserving $\mathbf{y}$, so that R moves the pair $(\mathbf{s}, \mathbf{x})$ to the pair $(\mathbf{t}, \mathbf{y})$ while preserving their lengths and angle. Verify too that R is proper orthogonal.

Problem (harder): Prove that every rotation in Euclidean 2- or 3-space is a product of two orthogonal reflections.. (The proof must ensure that both reflections exist.) How many suffice in Euclidean N-space?

§8: Changing to an Orthonormal Basis with Opposite Orientation

The vector $\mathbf{v}\times\mathbf{w} = \mathbf{v}^{¢}\mathbf{w}$ is sometimes called a *pseudo-vector* because of how an arbitrary change of orthonormal basis may affect it. For any orthogonal $Q = Q^{T–1}$ we shall find that

$$(Q\mathbf{v})^{¢}(Q\mathbf{w}) = Q\mathbf{v}^{¢}\mathbf{w}\cdot\det(Q)\ , \quad \text{or equivalently} \quad (Q\mathbf{v})^{¢} = Q\mathbf{v}^{¢}Q^T\cdot\det(Q)\ .$$

Of course $\det(Q) = \pm 1$; its appearance in the formula above is what deserves an explanation.

If $\det(Q) = +1$ then Q is a proper rotation and our geometrical intuition may well persuade us that rotating $\mathbf{v}$ and $\mathbf{w}$ together as a rigid body must rotate $\mathbf{v}^{¢}\mathbf{w}$ the same way, which is what the formula in question says. Otherwise $\det(Q) = –1$, in which case Q combines rotation and reflection; in this case the formula in question, in the form $(Q\mathbf{v})^{¢}Q = Q\mathbf{v}^{¢}\cdot\det(Q)$, will take some work to be confirmed. A comparatively simple proof is provided by …

David Meredith's Identity: $\mathrm{Adj}(L^T)\cdot\mathbf{v}^{\text{¢}} = (L\mathbf{v})^{\text{¢}}\cdot L$ for any 3-by-3 matrix L and vector $\mathbf{v}$ in Euclidean 3-space.

Into this identity substitute $L := Q$ and use $Q^T = Q^{-1}$ and $\mathrm{Adj}(Q^T) = (Q^T)^{-1}\cdot\det(Q^T) = Q\cdot\det(Q)$ to get the formula in question. What remains to be done is to prove the identity:

For *all* 3-vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ regarded as columns of a matrix $[\mathbf{v}, \mathbf{w}, \mathbf{u}]$, we find that

$$\det(L)\cdot\mathbf{u}^T\mathbf{v}^{\text{¢}}\mathbf{w} = \det(L)\cdot\det([\mathbf{v}, \mathbf{w}, \mathbf{u}]) = \det(L\cdot[\mathbf{v}, \mathbf{w}, \mathbf{u}]) = \det([L\mathbf{v}, L\mathbf{w}, L\mathbf{u}]) = (L\mathbf{u})^T(L\mathbf{v})^{\text{¢}}L\mathbf{w} = \mathbf{u}^TL^T(L\mathbf{v})^{\text{¢}}L\mathbf{w} .$$

Consequently $\det(L)\cdot\mathbf{v}^{\text{¢}} = L^T(L\mathbf{v})^{\text{¢}}L$. Into this equation substitute $\det(L)\cdot I = \det(L^T)\cdot I = L^T\cdot\mathrm{Adj}(L^T)$ when L^T is nonsingular to get first $L^T\cdot\mathrm{Adj}(L^T)\cdot\mathbf{v}^{\text{¢}} = L^T(L\mathbf{v})^{\text{¢}}L$, and then the desired identity. It is a polynomial equation in the elements of L and therefore valid also when L is singular. *Q.E.D.*

When $\det(Q) = -1$ the formulas just proved remind us that reflections reverse sense, changing right-handed triad $(\mathbf{v}, \mathbf{w}, \mathbf{v}^{\text{¢}}\mathbf{w})$ into left-handed triad $(Q\mathbf{v}, Q\mathbf{w}, Q\mathbf{v}^{\text{¢}}\mathbf{w})$, whereas $(Q\mathbf{v}, Q\mathbf{w}, (Q\mathbf{v})^{\text{¢}}(Q\mathbf{w}))$ is right-handed. (Look in a mirror to see why.) Consequently the last two triads' last elements must be oppositely directed.

Question: Why, when you look in a mirror, do you see *left* and *right* reversed there but not *up* and *down* ?

Answer: That's not what you see. (What you do see is described on the next page.)

§9: Applications of Cross-Products to Geometrical Problems

Cross-products $\mathbf{p}\times\mathbf{q}$, or $\mathbf{p}^{\text{¢}}\cdot\mathbf{q}$ in our matrix notation, figure prominently instead of determinants in neat textbook solutions of many commonplace geometrical problems. Our first example is …

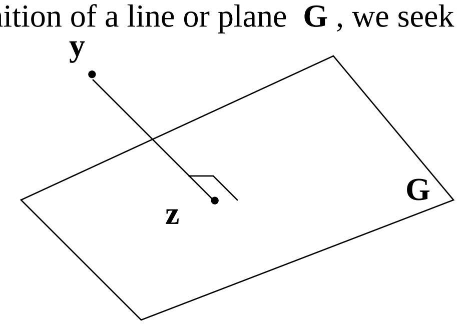
#**0.** Given the equations $\mathbf{p}^T\cdot\mathbf{x} = \pi$, $\mathbf{b}^T\cdot\mathbf{x} = ß$, $\mathbf{w}^T\cdot\mathbf{x} = \Omega$ of three planes, they intersect at point

$$\mathbf{z} = (\mathbf{b}^{\text{¢}}\cdot\mathbf{w}\cdot\pi + \mathbf{w}^{\text{¢}}\cdot\mathbf{p}\cdot ß + \mathbf{p}^{\text{¢}}\cdot\mathbf{b}\cdot\Omega)/(\mathbf{p}^T\cdot\mathbf{b}^{\text{¢}}\cdot\mathbf{w}) .$$

Neat formulas are more memorable and therefore more likely to appear in textbookss and be used by programmers than are ugly numerical algorithms like Gaussian Elimination even if the latter are numerically more stable. Gaussian Elimination is also faster than the foregoing formula; but a programmer can easily fix that by rewriting $\mathbf{z} = \big((\mathbf{b}^{\text{¢}}\cdot\mathbf{w})\cdot\pi + \mathbf{p}^{\text{¢}}\cdot(\mathbf{b}\cdot\Omega - \mathbf{w}\cdot ß) \big)/\big(\mathbf{p}^T\cdot(\mathbf{b}^{\text{¢}}\cdot\mathbf{w})\big)$ and reusing a common subexpression. Still, this formula is not so robust numerically as is Gaussian Elimination with pivotal exchanges.

Like Beauty, the neatness and speed of a formula lie in the eye of the beholding programmer sooner than does numerical stability. Roundoff doesn't figure in textbooks' formulas. The reader will not easily determine which are numerically unstable among the neat textbook formulas that solve the following eight commonplace geometrical problems each of the Nearest-Point kind:

Given a point $\mathbf{y}$ and the definition of a line or plane $\mathbf{G}$, we seek a point $\mathbf{z}$ in $\mathbf{G}$ nearest $\mathbf{y}$.



We expect the line segment joining $\mathbf{y}$ and $\mathbf{z}$ to stick out of $\mathbf{G}$ perpendicularly.

Answer to the *Question* above: What you actually see reversed in a mirror are *forward* and *back* .

If two formulas for $\mathbf{z}$ are offered below they suffer differently from rounding errors; the first formula suffers less than the second whenever $\|\mathbf{z}–\mathbf{y}\| << \|\mathbf{y}\|$ and the second less than the first whenever $\|\mathbf{z}\| << \|\mathbf{y}\|$. Unless parentheses indicate otherwise, associative products $\mathbf{A}·\mathbf{B}·\mathbf{C}$ should be evaluated in whichever order, $(\mathbf{A}·\mathbf{B})·\mathbf{C}$ or $\mathbf{A}·(\mathbf{B}·\mathbf{C})$, requires fewer arithmetic operations; doing so below tends to diminish roundoff too. An exercise for the diligent reader is to confirm the mathematical correctness of these formulas, even if roundoff may undermine their validity; casual readers will find their confirmations in §12.

"Numerically stable" solutions for these eight "linearly constrained least-squares" problems may be found in some works on numerical linear algebra; but no known stable solution is simply a rational formula like all those below.

#**1.** Given the equation $\mathbf{p}^T·\mathbf{x} = π$ of a plane Π, the point $\mathbf{z}$ in Π nearest $\mathbf{y}$ is

$$\mathbf{z} := \mathbf{y} – \mathbf{p}·(\mathbf{p}^T·\mathbf{y} – π)/\|\mathbf{p}\|^2 = (\mathbf{p}·π – \mathbf{p}^{¢}·\mathbf{p}^{¢}·\mathbf{y})/\|\mathbf{p}\|^2 .$$

#**2.** Given three points $\mathbf{u}–\mathbf{v}$, $\mathbf{u}$ and $\mathbf{u}+\mathbf{w}$ through which one plane Π passes, the point $\mathbf{z}$ in Π nearest $\mathbf{y}$ is $\mathbf{z} := \mathbf{y} – \mathbf{p}·\mathbf{p}^T·(\mathbf{y} – \mathbf{u})/\|\mathbf{p}\|^2 = \mathbf{u} – \mathbf{p}^{¢}·\mathbf{p}^{¢}·(\mathbf{y} – \mathbf{u})/\|\mathbf{p}\|^2$ wherein $\mathbf{p} := \mathbf{v}^{¢}·\mathbf{w}$.

#**3.** Given three points $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ through which one plane Π passes, the point $\mathbf{z}$ in Π nearest $\mathbf{y}$ is

$$\mathbf{z} := \mathbf{y} – \mathbf{p}·\mathbf{p}^T·(\mathbf{y} – \mathbf{u})/\|\mathbf{p}\|^2 = \mathbf{u} – \mathbf{p}^{¢}·\mathbf{p}^{¢}·(\mathbf{y} – \mathbf{u})/\|\mathbf{p}\|^2 \quad \text{wherein} \quad \mathbf{p} := (\mathbf{v} – \mathbf{u})^{¢}·(\mathbf{w} – \mathbf{u}) .$$

The order of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is permutable in each formula separately. To attenuate roundoff in $\mathbf{p}$ choose $\mathbf{u}$ to maximize $\|\mathbf{v} – \mathbf{w}\|$ roughly. (Why? See §11.) For $\mathbf{z}$ choose $\mathbf{u}$ to minimize $\|\mathbf{y} – \mathbf{u}\|$ in the first formula, $\|\mathbf{u}\|$ in the second.

#**4.** Given two points $\mathbf{u}$ and $\mathbf{u}+\mathbf{v}$ through which one line £ passes, the point $\mathbf{z}$ in £ nearest $\mathbf{y}$ is $\mathbf{z} := \mathbf{y} + \mathbf{v}^{¢}·\mathbf{v}^{¢}·(\mathbf{y} – \mathbf{u})/\|\mathbf{v}\|^2 = (\mathbf{v}·\mathbf{v}^T·\mathbf{y} – \mathbf{v}^{¢}·\mathbf{v}^{¢}·\mathbf{u})/\|\mathbf{v}\|^2 = \mathbf{u} + \mathbf{v}·\mathbf{v}^T·(\mathbf{y} – \mathbf{u})/\|\mathbf{v}\|^2$.

#**5.** Given two points $\mathbf{u}$ and $\mathbf{u}+\mathbf{v}$ through which one line £ passes, and two points $\mathbf{y}$ and $\mathbf{y}+\mathbf{w}$ through which another line ¥ passes, the point nearest £ in ¥ is $\mathbf{x} := \mathbf{y} + \mathbf{w}·\mathbf{v}^T·\mathbf{p}^{¢}·(\mathbf{y}–\mathbf{u})/\|\mathbf{p}\|^2$ wherein $\mathbf{p} = \mathbf{v}^{¢}·\mathbf{w}$. Nearest ¥ in £ is $\mathbf{z} := \mathbf{x} – \mathbf{p}·\mathbf{p}^T·(\mathbf{y}–\mathbf{u})/\|\mathbf{p}\|^2 = \mathbf{s} := \mathbf{u} + \mathbf{v}·\mathbf{w}^T·\mathbf{p}^{¢}·(\mathbf{y}–\mathbf{u})/\|\mathbf{p}\|^2$.

#**6.** Given two points $\mathbf{u}$ and $\mathbf{w}$ through which a line £ passes, the point $\mathbf{z}$ in £ nearest $\mathbf{y}$ is $\mathbf{z} := \mathbf{y} + \mathbf{v}^{¢}·\mathbf{v}^{¢}·(\mathbf{y} – \mathbf{u})/\|\mathbf{v}\|^2 = (\mathbf{v}·\mathbf{v}^T·\mathbf{y} – \mathbf{v}^{¢}·\mathbf{v}^{¢}·\mathbf{u})/\|\mathbf{v}\|^2 = \mathbf{u} + \mathbf{v}·\mathbf{v}^T·(\mathbf{y} – \mathbf{u})/\|\mathbf{v}\|^2$ wherein $\mathbf{v} := \mathbf{w} – \mathbf{u}$. Since $\mathbf{u}$ and $\mathbf{w}$ are permutable, choose $\mathbf{u}$ to minimize $\|\mathbf{y} – \mathbf{u}\|$ in the first and last formulas, and to minimize $\|\mathbf{u}\|$ in the middle formula, which is best if $\|\mathbf{z}\| << \|\mathbf{u}\|$ too.

#**7.** Given the two equations $\mathbf{p}^T·\mathbf{x} = π$ and $\mathbf{b}^T·\mathbf{x} = ß$ of a line £, the point $\mathbf{z}$ in £ nearest $\mathbf{y}$ is

$$\mathbf{z} := \mathbf{y} + \mathbf{v}^{¢}·(\mathbf{p}·(ß–\mathbf{b}^T·\mathbf{y}) – \mathbf{b}·(π–\mathbf{p}^T·\mathbf{y}))/\|\mathbf{v}\|^2 = (\mathbf{v}·\mathbf{v}^T·\mathbf{y} + \mathbf{v}^{¢}·(\mathbf{p}·ß–\mathbf{b}·π))/\|\mathbf{v}\|^2$$

wherein $\mathbf{v} := \mathbf{p}^{¢}·\mathbf{b}$. Of course we assume $\mathbf{v} \ne \mathbf{o}$ in order that £ be determined uniquely.

#**8.** Given three non-collinear points $\mathbf{u}-\mathbf{v}$, $\mathbf{u}$ and $\mathbf{u}+\mathbf{w}$ in Euclidean 3-space, the point

$$\mathbf{z} := \mathbf{u} + \tfrac{1}{2}(\mathbf{v}^{\varsigma}\mathbf{w})^{\varsigma}\cdot(\mathbf{v}\cdot\|\mathbf{w}\|^2 + \mathbf{w}\cdot\|\mathbf{v}\|^2)/\|\mathbf{v}^{\varsigma}\mathbf{w}\|^2 \;=\; \mathbf{u} + \tfrac{1}{2}(\|\mathbf{v}\|^2\cdot\mathbf{w}\mathbf{w}^T - \|\mathbf{w}\|^2\cdot\mathbf{v}\mathbf{v}^T)\cdot(\mathbf{v}+\mathbf{w})/\|\mathbf{v}^{\varsigma}\mathbf{w}\|^2$$

is the center of the circle through the three given points, and this circle's radius is

$$\tfrac{1}{2}\|\mathbf{v}\|\cdot\|\mathbf{w}\|\cdot\|\mathbf{v}+\mathbf{w}\|/\|\mathbf{v}^{\varsigma}\mathbf{w}\| .$$

When the given points are infinitesimally close neighbors on a smooth curve traced by $\mathbf{u} = \mathbf{u}(\tau)$ with non-vanishing velocity $\mathbf{u}' = \mathbf{u}'(\tau) := d\mathbf{u}(\tau)/d\tau$ and acceleration $\mathbf{u}'' = \mathbf{u}''(\tau) := d\mathbf{u}'(\tau)/d\tau$, the *Osculating Circle* that matches the curve's tangent and curvature at $\mathbf{u}$ is centered at the curve's *Center of Curvature* $\mathbf{c} := \mathbf{u} + \|\mathbf{u}'\|^2\cdot(\mathbf{u}'^{\varsigma}\cdot\mathbf{u}'')^{\varsigma}\cdot\mathbf{u}'/\|\mathbf{u}'^{\varsigma}\cdot\mathbf{u}''\|^2$. The circle's radius is the curve's *Radius of Curvature* $\|\mathbf{c}-\mathbf{u}\| = \|\mathbf{u}'\|^3/\|\mathbf{u}'^{\varsigma}\mathbf{u}''\|$. These formulas are derived in §12.

Problem #8 can arise in plane geometry. When $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ lie in a Euclidean 2-space, some of the formulas above simplify: The circle's center $\mathbf{z} := \mathbf{u} + \tfrac{1}{2}J\cdot(\mathbf{v}\cdot\|\mathbf{w}\|^2 + \mathbf{w}\cdot\|\mathbf{v}\|^2)/(\mathbf{w}^TJ\mathbf{v})$ and its radius $\|\mathbf{c}-\mathbf{u}\| = \tfrac{1}{2}\|\mathbf{v}\|\cdot\|\mathbf{w}\|\cdot\|\mathbf{v}+\mathbf{w}\|/|\mathbf{w}^TJ\mathbf{v}|$; the center of curvature $\mathbf{c} := \mathbf{u} + \|\mathbf{u}'\|^2J\mathbf{u}'/(\mathbf{u}''^TJ\mathbf{u}')$ and the curve's radius of curvature is $\|\mathbf{c}-\mathbf{u}\| = \|\mathbf{u}'\|^3/|\mathbf{u}''^TJ\mathbf{u}'|$.

§10: An Unfortunate Numerical Example

It is not at all obvious that formula #7, say, is numerically unstable. In fact all figures carried can be lost if a few too few are carried. Try this data all stored exactly as 4-byte `floats` :

$\mathbf{p}^T = [\, 38006,\ 23489,\ 14517 \,]$, $\pi = 8972$, $\mathbf{b}^T = [\, 23489,\ 14517,\ 8972 \,]$, $ß = 5545$, and $\mathbf{y}^T = [\, 1,\ -1,\ 1 \,]$.

This data defines £ as the intersection of two nearly parallel planes, so tiny changes in data can alter £ and $\mathbf{z}$ drastically. More troublesome numerically are the many correlated appearances of the data $\mathbf{p}$ and $\mathbf{b}$ in the formulas for $\mathbf{z}$ in problem #7. Though mathematically crucial, these correlations can be ruined by roundoff. Evaluating both formulas above for $\mathbf{z}$ naively in `float` arithmetic yields $\mathbf{z}_1{}^T = [\, 1,\ 1,\ -1 \,]$ and $\mathbf{z}_2{}^T = [\, 1,\ 1,\ -1.5 \,]$; but $\mathbf{z}_1$ lies farther from both planes than 0.8, and $\mathbf{z}_2$ lies farther from them than 0.6. These gaps cannot be attributed to end-figure "errors" in the given data which would shift £ and $\mathbf{z}$ but not separate them.

***This naive arithmetic produces geometrically impossible results*.**

The correct point $\mathbf{z}^T = [\, 1/3,\ 2/3,\ -4/3 \,]$ is computed correctly rounded when all intermediate results (sub-expressions and local variables) are evaluated in `double` before $\mathbf{z}$ is rounded back to `float` . Naively computed $\mathbf{z}_1$ and $\mathbf{z}_2$ are not so far from $\mathbf{z}$ as to be obviously wrong if $\mathbf{z}$ were unknown, but but they are too wrong to be acceptable for most purposes.

This unfortunate example exemplifies behavior that occurs surprisingly often: Unless evaluated extra-precisely, too many neat formulas can be intolerably more inaccurate than the data deserve.

WHY ?

§11: Bilinear forms vulnerable to roundoff followed by cancellation occur frequently:

Scalar products: $\mathbf{p}\bullet\mathbf{b} = \mathbf{p}^T\cdot\mathbf{b} = p_1 \cdot b_1 + p_2 \cdot b_2 + p_3 \cdot b_3$.

Linear combinations: $\mathbf{p}\cdot ß - \mathbf{b}\cdot\pi = \begin{bmatrix} p_1 \cdot \beta - b_1 \cdot \pi \\ p_2 \cdot \beta - b_2 \cdot \pi \\ p_3 \cdot \beta - b_3 \cdot \pi \end{bmatrix}$.

Cross products: $\mathbf{p}\times\mathbf{b} = \mathbf{p}^{¢}\cdot\mathbf{b} = \begin{bmatrix} p_2 \cdot b_3 - p_3 \cdot b_2 \\ p_3 \cdot b_1 - p_1 \cdot b_3 \\ p_1 \cdot b_2 - p_2 \cdot b_1 \end{bmatrix}$.

These entities are *geometrically redundant*; they are so correlated that $(\mathbf{p}\cdot ß - \mathbf{b}\cdot\pi)\bullet(\mathbf{p}\times\mathbf{b}) = 0$ for *all* data $\{\mathbf{p}, \pi, \mathbf{b}, ß\}$. Even if data are "accurate" to few sig. digits and computed entities to fewer, their geometrical redundancy must be conserved as accurately as possible. We can tolerate slightly inaccurate results interpretable as realizable geometrical objects slightly different from our original intent, but not geometrically impossible objects like a $\mathbf{p}\times\mathbf{b}$ too far from orthogonal to $\mathbf{p}$ and $\mathbf{b}$ because of roundoff. Suppose ε is the roundoff threshold, meaning that sums, differences and products are computed accurately within a factor $1\pm\varepsilon$. For instance, $\varepsilon = 5/10^{10}$ for arithmetic rounded to 10 sig. dec. Then the angle between the desired cross product $\mathbf{p}\times\mathbf{b}$ and its computed version will be typically about $\pm\varepsilon/\sin(\angle(\mathbf{p}, \mathbf{b}))$ and is proved never to get much bigger in §13. This shows how roundoff degrades $\mathbf{p}\times\mathbf{b}$ as $\mathbf{p}$ and $\mathbf{b}$ approach (anti)parallelism and, in view of the *Sine Law of Triangles*, justifies advice about $(\mathbf{v} - \mathbf{u})^{¢}\cdot(\mathbf{w} - \mathbf{u})$ in problem #3.

Therefore these bilinear forms and other matrix products should be computed carrying somewhat more precision than in the data, thereby preserving geometrical redundancy despite "losses" of several digits to cancellation. At any precision, prolonged chains of computation risk losing geometrical redundancy. The wider is the precision, the longer is that loss postponed and the more often prevented, provided that extra-precise arithmetic does not run intolerably slowly.

And extra precision usually costs less than error-analysis.

This is not said to disparage error-analysis; it is always the right thing to do if you know how and have the time. But to know how you have to take advanced classes in numerical analysis since elementary classes hardly ever cover error-analysis well enough to be useful. To spend enough time you have to believe that the results being (in)validated by error-analysis are worth the time.

For extensive discussions of these and similar computational issues see …

"Marketing versus Mathematics" `www.cs.berkeley.edu/~wkahan/MktgMath.pdf`
"How Java's Floating-Point Hurts Everybody Everywhere" *ibid.* … `/JAVAhurt.pdf`
"Miscalculating Area and Angles of a Needle-like Triangle" *ibid.* … `/Triangle.pdf`
"MATLAB's Loss is Nobody's Gain" *ibid.* … `/MxMulEps.pdf`
"How Futile are Mindless Assessments of Roundoff in Floating-Point Computation ?" *ibid.* … `/Mindless.pdf`
Prof. Jonathan Shewchuk's web page, starting with `www.cs.berkeley.edu/~jrs/meshpapers/robnotes.ps.gz`
…..

A good book about Error-Analysis is N.J. Higham's *Accuracy and Stability of Numerical Algorithms* 2d ed. (2002) Soc. Indust. & Appl. Math., Philadelphia; but it is about 700 pages long.

§12: Confirmations of the Eight Formulas in § 9 :

#**1.** Given the equation $\mathbf{p}^T\cdot\mathbf{x} = \pi$ of a plane $\prod$, the point $\mathbf{z}$ in $\prod$ nearest $\mathbf{y}$ is

$$\mathbf{z} := \mathbf{y} - \mathbf{p}\cdot(\mathbf{p}^T\cdot\mathbf{y} - \pi)/\|\mathbf{p}\|^2 \;=\mathrm{s} :=\; (\,\mathbf{p}\cdot\pi - \mathbf{p}^{\text{¢}}\cdot\mathbf{p}^{\text{¢}}\cdot\mathbf{y}\,)/\|\mathbf{p}\|^2 \,.$$

To confirm these formulas we must verify that $\mathbf{p}^T\cdot\mathbf{z} = \pi$, putting $\mathbf{z}$ in $\prod$, and that $\mathbf{p}$ is (anti)parallel to $\mathbf{z}\mathbf{–}\mathbf{y}$ so that it is perpendicular to $\prod$. And then we must verify that $\mathbf{z} = \mathbf{s}$. Only the last is unobvious:

$$(\mathbf{z}-\mathbf{s})\cdot\|\mathbf{p}\|^2 = \mathbf{y}\cdot\|\mathbf{p}\|^2 - \mathbf{p}\cdot(\mathbf{p}^T\cdot\mathbf{y} - \pi) - (\,\mathbf{p}\cdot\pi - \mathbf{p}^{\text{¢}}\cdot\mathbf{p}^{\text{¢}}\cdot\mathbf{y}\,) = \mathbf{o} \;\text{ since }\; (\mathbf{p}^{\text{¢}})^2 = \mathbf{p}\cdot\mathbf{p}^T - \|\mathbf{p}\|^2\cdot\mathrm{I}\,.$$

#**2.** Given three points $\mathbf{u}-\mathbf{v}$, $\mathbf{u}$ and $\mathbf{u}+\mathbf{w}$ through which one plane $\prod$ passes, the point $\mathbf{z}$ in $\prod$ nearest $\mathbf{y}$ is $\mathbf{z} := \mathbf{y} - \mathbf{p}\cdot\mathbf{p}^T\cdot(\mathbf{y} - \mathbf{u})/\|\mathbf{p}\|^2 \;=\mathrm{s} :=\; \mathbf{u} - \mathbf{p}^{\text{¢}}\cdot\mathbf{p}^{\text{¢}}\cdot(\mathbf{y} - \mathbf{u})/\|\mathbf{p}\|^2$ wherein $\mathbf{p} := \mathbf{v}^{\text{¢}}\cdot\mathbf{w}$. These formulas follow from problem **1** because the plane's equation is $\mathbf{p}^T\cdot\mathbf{x} = \pi := \mathbf{p}^T\cdot\mathbf{u}$.

#**3.** Given three points $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ through which one plane $\prod$ passes, the point $\mathbf{z}$ in $\prod$ nearest $\mathbf{y}$ is

$$\mathbf{z} := \mathbf{y} - \mathbf{p}\cdot\mathbf{p}^T\cdot(\mathbf{y} - \mathbf{u})/\|\mathbf{p}\|^2 \;=\mathrm{s} :=\; \mathbf{u} - \mathbf{p}^{\text{¢}}\cdot\mathbf{p}^{\text{¢}}\cdot(\mathbf{y} - \mathbf{u})/\|\mathbf{p}\|^2 \quad\text{wherein}\quad \mathbf{p} := (\mathbf{v} - \mathbf{u})^{\text{¢}}\cdot(\mathbf{w} - \mathbf{u})\,.$$

These formulas follow from problem #2 after its $\mathbf{v}$ and $\mathbf{w}$ are replaced by $\mathbf{u}-\mathbf{v}$ and $\mathbf{w}-\mathbf{u}$.

#**4.** Given two points $\mathbf{u}$ and $\mathbf{u}+\mathbf{v}$ through which one line £ passes, the point $\mathbf{z}$ in £ nearest $\mathbf{y}$ is $\mathbf{z} := \mathbf{y} + \mathbf{v}^{\text{¢}}\cdot\mathbf{v}^{\text{¢}}\cdot(\mathbf{y} - \mathbf{u})/\|\mathbf{v}\|^2 \;=\mathrm{s} :=\; (\,\mathbf{v}\cdot\mathbf{v}^T\cdot\mathbf{y} - \mathbf{v}^{\text{¢}}\cdot\mathbf{v}^{\text{¢}}\cdot\mathbf{u}\,)/\|\mathbf{v}\|^2 \;= \mathbf{t} :=\; \mathbf{u} + \mathbf{v}\cdot\mathbf{v}^T\cdot(\mathbf{y} - \mathbf{u})/\|\mathbf{v}\|^2$. To confirm these formulas we must verify that $\mathbf{z}-\mathbf{u}$ is a scalar multiple of $\mathbf{v}$, which places $\mathbf{z}$ on £, and that $\mathbf{v}^T(\mathbf{z}-\mathbf{y}) = 0$ so that $\mathbf{z}-\mathbf{y}$ is perpendicular to £. Since $(\mathbf{v}^{\text{¢}})^2 = \mathbf{v}\cdot\mathbf{v}^T - \|\mathbf{v}\|^2\cdot\mathrm{I}$, we find that $\mathbf{z}-\mathbf{u} = \mathbf{v}\cdot\mathbf{v}^T\cdot(\mathbf{y}-\mathbf{u})/\|\mathbf{v}\|^2 = \mathbf{t}-\mathbf{u}$ *is* a scalar multiple of $\mathbf{v}$ and, incidentally, $\mathbf{z} = \mathbf{t}$. And $\mathbf{v}^T(\mathbf{z}-\mathbf{y}) = 0$ follows from $\mathbf{v}^T\mathbf{v}^{\text{¢}} = \mathbf{o}^T$. Finally $\mathbf{s} = \mathbf{t}$ follows from the expansion of $(\mathbf{v}^{\text{¢}})^2$.

#**5.** Given two points $\mathbf{u}$ and $\mathbf{u}+\mathbf{v}$ through which one line £ passes, and two points $\mathbf{y}$ and $\mathbf{y}+\mathbf{w}$ through which another line ¥ passes, the point nearest £ in ¥ is $\mathbf{x} := \mathbf{y} + \mathbf{w}\cdot\mathbf{v}^T\cdot\mathbf{p}^{\text{¢}}\cdot(\mathbf{y}-\mathbf{u})/\|\mathbf{p}\|^2$ wherein $\mathbf{p} = \mathbf{v}^{\text{¢}}\cdot\mathbf{w}$. Nearest ¥ in £ is $\mathbf{z} := \mathbf{x} - \mathbf{p}\cdot\mathbf{p}^T\cdot(\mathbf{y}-\mathbf{u})/\|\mathbf{p}\|^2 = \mathbf{s} := \mathbf{u} + \mathbf{v}\cdot\mathbf{w}^T\cdot\mathbf{p}^{\text{¢}}\cdot(\mathbf{y}-\mathbf{u})/\|\mathbf{p}\|^2$. To confirm these formulas, we confirm first that $\mathbf{x}$ lies in ¥ because $\mathbf{x}-\mathbf{y} = \mathbf{w}\cdot(\text{scalar})$, and that $\mathbf{s}$ lies in £ because $\mathbf{s}-\mathbf{u} = \mathbf{v}\cdot(\text{scalar})$. Then $\mathbf{z} = \mathbf{s}$ because

$$\begin{aligned}\|\mathbf{p}\|^2\cdot(\mathbf{z}-\mathbf{s}) &= (\|\mathbf{p}\|^2\cdot\mathrm{I} - \mathbf{p}\cdot\mathbf{p}^T + \mathbf{w}\cdot\mathbf{v}^T\cdot\mathbf{p}^{\text{¢}} - \mathbf{v}\cdot\mathbf{w}^T\cdot\mathbf{p}^{\text{¢}})(\mathbf{y}-\mathbf{u}) \quad \ldots\ \text{recall}\ (\mathbf{p}^{\text{ç}})^2 = \mathbf{p}\cdot\mathbf{p}^T - \|\mathbf{p}\|^2\cdot\mathrm{I}\\ &= (-\mathbf{p}^{\text{¢}} + \mathbf{w}\cdot\mathbf{v}^T - \mathbf{v}\cdot\mathbf{w}^T)\cdot\mathbf{p}^{\text{¢}}\cdot(\mathbf{y}-\mathbf{u}) \;=\; \mathbf{O}\cdot\mathbf{p}^{\text{¢}}\cdot(\mathbf{y}-\mathbf{u}) = \mathbf{o}\end{aligned}$$

in view of the triple cross-product formula $\mathbf{p}^{\text{¢}} = (\mathbf{v}^{\text{¢}}\mathbf{w})^{\text{¢}} = \mathbf{w}\cdot\mathbf{v}^T - \mathbf{v}\cdot\mathbf{w}^T$ in §3 and §5. Finally, $\mathbf{z}-\mathbf{x} = \mathbf{p}\cdot(\text{scalar})$ is perpendicular to both lines ¥ and £, so $\mathbf{z}$ and $\mathbf{x}$ are nearest each other.

Lest binocular vision's depth perception and range-finding falter when £ and ¥ almost intersect, $(\mathbf{x}+\mathbf{z})/2 = \big(\,\mathbf{u}+\mathbf{y} + (\mathbf{w}\cdot\mathbf{v}^T + \mathbf{v}\cdot\mathbf{w}^T)\cdot\mathbf{p}^{\text{¢}}\cdot(\mathbf{y}-\mathbf{u})/\|\mathbf{p}\|^2\,\big)/2$ is the point nearest their near-intersection.

#**6.** Given two points $\mathbf{u}$ and $\mathbf{w}$ through which a line £ passes, the point $\mathbf{z}$ in £ nearest $\mathbf{y}$ is $\mathbf{z} = \mathbf{y} + \mathbf{v}^{\text{¢}}\cdot\mathbf{v}^{\text{¢}}\cdot(\mathbf{y} - \mathbf{u})/\|\mathbf{v}\|^2 \;=\; (\,\mathbf{v}\cdot\mathbf{v}^T\cdot\mathbf{y} - \mathbf{v}^{\text{¢}}\cdot\mathbf{v}^{\text{¢}}\cdot\mathbf{u}\,)/\|\mathbf{v}\|^2 \;=\; \mathbf{u} + \mathbf{v}\cdot\mathbf{v}^T\cdot(\mathbf{y} - \mathbf{u})/\|\mathbf{v}\|^2$ wherein $\mathbf{v} = \mathbf{w} - \mathbf{u}$. These formulas follow from problem #4.

#7. Given the two equations $\mathbf{p}^T\cdot\mathbf{x} = \pi$ and $\mathbf{b}^T\cdot\mathbf{x} = \text{ß}$ of a line $\pounds$, the point $\mathbf{z}$ in $\pounds$ nearest $\mathbf{y}$ is

$$\mathbf{z} := \mathbf{y} + \mathbf{v}^{\text{¢}}\cdot(\,\mathbf{p}\cdot(\text{ß}-\mathbf{b}^T\cdot\mathbf{y}) - \mathbf{b}\cdot(\pi-\mathbf{p}^T\cdot\mathbf{y})\,)/\|\mathbf{v}\|^2 \;=\; \mathbf{s} := (\,\mathbf{v}\cdot\mathbf{v}^T\cdot\mathbf{y} + \mathbf{v}^{\text{¢}}\cdot(\mathbf{p}\cdot\text{ß}-\mathbf{b}\cdot\pi)\,)/\|\mathbf{v}\|^2$$

wherein $\mathbf{v} = \mathbf{p}^{\text{¢}}\cdot\mathbf{b} \neq \mathbf{o}$. This $\mathbf{v}$ is parallel to $\pounds$ because it is perpendicular to the normals of both planes that intersect in $\pounds$. To confirm that $\mathbf{z}$ is the point in $\pounds$ nearest $\mathbf{y}$ we must verify that $\mathbf{z}$ lies in both those planes and that $\mathbf{v}$ is perpendicular to $\mathbf{z}-\mathbf{y}$. In other words, we must verify that $\mathbf{p}^T\cdot\mathbf{z} = \pi$, $\mathbf{b}^T\cdot\mathbf{z} = \text{ß}$, $\mathbf{v}^T\cdot(\mathbf{z}-\mathbf{y}) = 0$ and $\mathbf{s} = \mathbf{z}$. This was done with DERIVE™, a computerized algebra system, faster than the following dozen lines of a manual verification can be written out:

$$\mathbf{v}^T\cdot(\mathbf{z}-\mathbf{y}) = \mathbf{v}^T\cdot\mathbf{v}^{\text{¢}}\cdot(\,\mathbf{p}\cdot(\text{ß}-\mathbf{b}^T\cdot\mathbf{y}) - \mathbf{b}\cdot(\pi-\mathbf{p}^T\cdot\mathbf{y})\,)/\|\mathbf{v}\|^2 = 0 \text{ because } \mathbf{v}^T\cdot\mathbf{v}^{\text{¢}} = \mathbf{o}^T .$$

$$\begin{aligned}
\mathbf{p}^T\cdot\mathbf{z} - \pi &= \mathbf{p}^T\cdot\mathbf{y} - \pi + \mathbf{p}^T\cdot\mathbf{v}^{\text{¢}}\cdot(\,\mathbf{p}\cdot(\text{ß}-\mathbf{b}^T\cdot\mathbf{y}) - \mathbf{b}\cdot(\pi-\mathbf{p}^T\cdot\mathbf{y})\,)/\|\mathbf{v}\|^2\\
&= \mathbf{p}^T\cdot\mathbf{y} - \pi - \mathbf{p}^T\cdot\mathbf{v}^{\text{¢}}\cdot\mathbf{b}\cdot(\pi-\mathbf{p}^T\cdot\mathbf{y})/\|\mathbf{v}\|^2 \quad \text{because } \mathbf{p}^T\cdot\mathbf{v}^{\text{¢}}\cdot\mathbf{p} = 0 ,\\
&= \mathbf{p}^T\cdot\mathbf{y} - \pi + \mathbf{v}^T\cdot\mathbf{p}^{\text{¢}}\cdot\mathbf{b}\cdot(\pi-\mathbf{p}^T\cdot\mathbf{y})/\|\mathbf{v}\|^2 \quad \text{because } \mathbf{p}^T\cdot\mathbf{v}^{\text{¢}}\cdot\mathbf{b} = -\mathbf{v}^T\cdot\mathbf{p}^{\text{¢}}\cdot\mathbf{b} ,\\
&= \mathbf{p}^T\cdot\mathbf{y} - \pi + \mathbf{v}^T\cdot\mathbf{v}\cdot(\pi-\mathbf{p}^T\cdot\mathbf{y})/\|\mathbf{v}\|^2 = 0 .
\end{aligned}$$

$$\begin{aligned}
\mathbf{b}^T\cdot\mathbf{z} - \text{ß} &= \mathbf{b}^T\cdot\mathbf{y} - \text{ß} + \mathbf{b}^T\cdot\mathbf{v}^{\text{¢}}\cdot(\,\mathbf{p}\cdot(\text{ß}-\mathbf{b}^T\cdot\mathbf{y}) - \mathbf{b}\cdot(\pi-\mathbf{p}^T\cdot\mathbf{y})\,)/\|\mathbf{v}\|^2\\
&= \mathbf{b}^T\cdot\mathbf{y} - \text{ß} + \mathbf{b}^T\cdot\mathbf{v}^{\text{¢}}\cdot\mathbf{p}\cdot(\text{ß}-\mathbf{b}^T\cdot\mathbf{y})/\|\mathbf{v}\|^2 \quad \text{because } \mathbf{b}^T\cdot\mathbf{v}^{\text{¢}}\cdot\mathbf{b} = 0 ,\\
&= \mathbf{b}^T\cdot\mathbf{y} - \text{ß} - \mathbf{v}^T\cdot\mathbf{b}^{\text{¢}}\cdot\mathbf{p}\cdot(\text{ß}-\mathbf{b}^T\cdot\mathbf{y})/\|\mathbf{v}\|^2 \quad \text{because } \mathbf{b}^T\cdot\mathbf{v}^{\text{¢}}\cdot\mathbf{p} = -\mathbf{v}^T\cdot\mathbf{b}^{\text{¢}}\cdot\mathbf{p} ,\\
&= \mathbf{b}^T\cdot\mathbf{y} - \text{ß} + \mathbf{v}^T\cdot\mathbf{v}\cdot(\text{ß}-\mathbf{b}^T\cdot\mathbf{y})/\|\mathbf{v}\|^2 = 0 .
\end{aligned}$$

$$\begin{aligned}
(\mathbf{z}-\mathbf{s})\cdot\|\mathbf{v}\|^2 &= (\mathbf{v}^T\cdot\mathbf{v}\cdot I - \mathbf{v}\cdot\mathbf{v}^T)\cdot\mathbf{y} + \mathbf{v}^{\text{¢}}\cdot(\,\mathbf{p}\cdot(\text{ß}-\mathbf{b}^T\cdot\mathbf{y}) - \mathbf{b}\cdot(\pi-\mathbf{p}^T\cdot\mathbf{y})\,) - \mathbf{v}^{\text{¢}}\cdot(\mathbf{p}\cdot\text{ß}-\mathbf{b}\cdot\pi)\\
&= -(\mathbf{v}^{\text{¢}})^2\cdot\mathbf{y} + \mathbf{v}^{\text{¢}}\cdot(\,\mathbf{p}\cdot(\text{ß}-\mathbf{b}^T\cdot\mathbf{y}) - \mathbf{b}\cdot(\pi-\mathbf{p}^T\cdot\mathbf{y}) - (\mathbf{p}\cdot\text{ß}-\mathbf{b}\cdot\pi)\,)\\
&= \mathbf{v}^{\text{¢}}\cdot(\,-\mathbf{v}^{\text{¢}} - \mathbf{p}\cdot\mathbf{b}^T + \mathbf{b}\cdot\mathbf{p}^T\,)\cdot\mathbf{y} = \mathbf{v}^{\text{¢}}\cdot(\,-\mathbf{v}^{\text{¢}} + (\mathbf{p}^{\text{¢}}\cdot\mathbf{b})^{\text{¢}}\,)\cdot\mathbf{y} = \mathbf{o} .
\end{aligned}$$

#8. Given three non-collinear points $\mathbf{u}-\mathbf{v}$, $\mathbf{u}$ and $\mathbf{u}+\mathbf{w}$ in Euclidean 3-space, the center $\mathbf{z}$ of the circle through the given points is $\mathbf{z} := \mathbf{u} + \frac{1}{2}(\|\mathbf{v}\|^2\cdot\mathbf{w}\mathbf{w}^T - \|\mathbf{w}\|^2\cdot\mathbf{v}\mathbf{v}^T)\cdot(\mathbf{v}+\mathbf{w})/\|\mathbf{v}^{\varsigma}\mathbf{w}\|^2$, and the circle's radius is $\frac{1}{2}\|\mathbf{v}\|\cdot\|\mathbf{w}\|\cdot\|\mathbf{v}+\mathbf{w}\|/\|\mathbf{v}^{\varsigma}\mathbf{w}\|$. To prove these formulas this problem will be reduced to a problem solved previously.

Among points equidistant from the three given points, center $\mathbf{z}$ is nearest them. To simplify its derivation shift the origin to $\mathbf{u}$ temporarily. Now "equidistant" implies $\|\mathbf{v}+\mathbf{z}\|^2 = \|\mathbf{z}\|^2 = \|\mathbf{w}-\mathbf{z}\|^2$ which boils down to two linear equations $\mathbf{v}^T\mathbf{z} = -\|\mathbf{v}\|^2/2$ and $\mathbf{w}^T\mathbf{z} = \|\mathbf{w}\|^2/2$, restricting $\mathbf{z}$ to a straight line. This reduces problem #8 to an instance of problem #7 whose solution becomes $\mathbf{z} = \mathbf{u} + \frac{1}{2}(\mathbf{v}^{\varsigma}\mathbf{w})^{\varsigma}\cdot(\mathbf{v}\cdot\|\mathbf{w}\|^2 + \mathbf{w}\cdot\|\mathbf{v}\|^2)/\|\mathbf{v}^{\varsigma}\mathbf{w}\|^2 = \mathbf{u} + \frac{1}{2}(\|\mathbf{v}\|^2\cdot\mathbf{w}\mathbf{w}^T - \|\mathbf{w}\|^2\cdot\mathbf{v}\mathbf{v}^T)\cdot(\mathbf{v}+\mathbf{w})/\|\mathbf{v}^{\varsigma}\mathbf{w}\|^2$ after some simplification and restoration of the origin. Note that this formula makes $\mathbf{z}$ coplanar with the three given points, as must be expected. Now, $\mathbf{z}$ is the center of a circle whose radius is

$$\begin{aligned}
\|\mathbf{z}-\mathbf{u}\| &= \tfrac{1}{2}\|(\|\mathbf{v}\|^2\cdot\mathbf{w}\mathbf{w}^T - \|\mathbf{w}\|^2\cdot\mathbf{v}\mathbf{v}^T)\cdot(\mathbf{v}+\mathbf{w})\|/\|\mathbf{v}^{\varsigma}\mathbf{w}\|^2\\
&= \tfrac{1}{2}\|\,\mathbf{w}\cdot\|\mathbf{v}\|^2\cdot(\|\mathbf{w}\|^2 + \mathbf{v}^T\mathbf{w}) - \mathbf{v}\cdot\|\mathbf{w}\|^2\cdot(\|\mathbf{v}\|^2 + \mathbf{v}^T\mathbf{w})\,\|/\|\mathbf{v}^{\varsigma}\mathbf{w}\|^2 = \dots\\
&= \tfrac{1}{2}\|\mathbf{v}\|\cdot\|\mathbf{w}\|\cdot\|\mathbf{v}+\mathbf{w}\|\cdot\sqrt{(\|\mathbf{v}\|^2\cdot\|\mathbf{w}\|^2 - (\mathbf{v}^T\mathbf{w})^2)}/\|\mathbf{v}^{\varsigma}\mathbf{w}\|^2 = \tfrac{1}{2}\|\mathbf{v}\|\cdot\|\mathbf{w}\|\cdot\|\mathbf{v}+\mathbf{w}\|/\|\mathbf{v}^{\varsigma}\mathbf{w}\|
\end{aligned}$$

as claimed.

Next let $\mathbf{u} := \mathbf{u}(\tau)$ trace a smooth curve with non-vanishing velocity $\mathbf{u'} := \mathbf{u'}(\tau) := d\mathbf{u}(\tau)/d\tau$ and acceleration $\mathbf{u''} := \mathbf{u''}(\tau) := d\mathbf{u'}(\tau)/d\tau$. The Taylor series $\mathbf{u}(\tau+\theta) = \mathbf{u} + \theta\cdot\mathbf{u'} + \theta^2\cdot\mathbf{u''}/2 + \dots$ determines $\mathbf{v} := \mathbf{u}(\tau) - \mathbf{u}(\tau-\phi) = \phi\cdot\mathbf{u'} - \phi^2\cdot\mathbf{u''}/2 + \dots$ and $\mathbf{w} := \mathbf{u}(\tau+\theta) - \mathbf{u}(\tau)$ in the foregoing first formula for $\mathbf{z}$ whose limit, as $\theta \to 0$ and $\phi \to 0$, turns into the *Center of Curvature*

$$\mathbf{c} = \mathbf{u} - \|\mathbf{u'}\|^2\cdot\mathbf{u'}^{\varsigma}\cdot(\mathbf{u'}^{\varsigma}\cdot\mathbf{u''})/\|\mathbf{u'}^{\varsigma}\cdot\mathbf{u''}\|^2 .$$

And then because $\mathbf{u'}^T\cdot(\mathbf{u'}^{\varsigma}\cdot\mathbf{u''}) = 0$, the *Radius of Curvature* $\|\mathbf{c}-\mathbf{u}\| = \|\mathbf{u'}\|^3/\|\mathbf{u'}^{\varsigma}\mathbf{u''}\|$.

§13: Rounding Error-Bounds for Angles

The angle between the desired cross product $\mathbf{p}\times\mathbf{b}$ and its computed version is alleged in §11 to never exceed $\varepsilon/|\sin(\angle(\mathbf{p}, \mathbf{b}))|$ much. Here ε is the roundoff threshold for individual arithmetic operations. This means that executing an assignment statement " $x := y\cdot z$ " actually computes and stores some rounded value $x := (1\pm\varepsilon)\cdot y\cdot z$, which is how an unknown number between $(1-\varepsilon)\cdot y\cdot z$ and $(1+\varepsilon)\cdot y\cdot z$ shall be described. Likewise " $x := y-z$ " actually stores a number $x = (1\pm\varepsilon)\cdot(y-z)$. In special circumstances more than this can be said about x ; for instance, if $1/2 \le y/z \le 2$ then " $x := y-z$ " actually stores $x = y-z$ exactly on almost all today's computers.

Problem: Perhaps aided by a calculator, explore and then confirm the last assertion, and then find examples that would violate it if " 1/2 " were diminished or " 2 " increased.

In any event, ε is very tiny. $\varepsilon = 1/2^{24} \approx 5.96/10^8$ for 4-byte wide `floats`. For the 8-byte wide floating-point arithmetic used by MATLAB, $\varepsilon = 1/2^{53} =$ `eps/2` $\approx 1.11/10^{16}$. Consequently terms of order ε^2 will be ignored below.

A strict version of §11's allegation is that, if $\mathbf{w} \ne \mathbf{o}$ is the column vector computed for $\mathbf{p}\times\mathbf{b} \ne \mathbf{o}$ from $\mathbf{p}$ and $\mathbf{b}$ using floating-point arithmetic whose roundoff threshold is ε , then there is some constant $\mu \le 4/\sqrt{3} \approx 2.3094$ for which $|\sin(\angle(\mathbf{w}, \mathbf{p}\times\mathbf{b}))| \le \mu\cdot\varepsilon/|\sin(\angle(\mathbf{p}, \mathbf{b}))|$. A proof follows:

To reduce the strain on aged eyes, subscripts and superscripts will be avoided wherever possible. Set column-vectors $\mathbf{p} := [x, y, z]^T$ and $\mathbf{b} := [e, f, g]^T$, so $\mathbf{p}\times\mathbf{b} = [y\cdot g-z\cdot f,\ z\cdot e-x\cdot g,\ x\cdot f-y\cdot e]^T$. The computed cross-product $\mathbf{w}$ has elements of which the first, $(1\pm\varepsilon)\cdot((1\pm\varepsilon)\cdot y\cdot g - (1\pm\varepsilon)\cdot z\cdot f)$, is typical. Ignoring second-order terms in ε , this amounts to $y\cdot g-z\cdot f \pm 2\varepsilon\cdot(|y\cdot g| + |z\cdot f|)$ at worst. Consequently $|\mathbf{w} - \mathbf{p}\times\mathbf{b}| \le 2\varepsilon\cdot|\mathbf{p}^{¢}|\cdot|\mathbf{b}|$ elementwise; here $\mathbf{p}^{¢}$ is the skew matrix that produces $\mathbf{p}^{¢}\cdot\mathbf{b} = \mathbf{p}\times\mathbf{b}$. Therefore, using the Euclidean norm $\|\mathbf{v}\| := \sqrt{(\mathbf{v}^T\mathbf{v})}$, we conclude that

$$\|\mathbf{w} - \mathbf{p}\times\mathbf{b}\|^2 \le 4\varepsilon^2\cdot|\mathbf{b}|^T\cdot|\mathbf{p}^{¢}|^2\cdot|\mathbf{b}| \le 4\varepsilon^2\cdot\|\mathbf{b}\|^2\cdot(\text{ the biggest eigenvalue of } |\mathbf{p}^{¢}|\,)^2 ,$$

since $|\mathbf{p}^{¢}|$ is a real symmetric matrix. Its *Characteristic Polynomial* turns out to be

$$f(\lambda) := \det(\lambda I - |\mathbf{p}^{¢}|\,) = \lambda^3 - \|\mathbf{p}\|^2\lambda - 2|x\cdot y\cdot z| \quad \text{wherein } \|\mathbf{p}\|^2 = x^2 + y^2 + z^2 > 0 .$$

To locate its zeros, the eigenvalues of $|\mathbf{p}^{¢}|$, we shall repeatedly use the *Arithmetic-Geometric Means Inequality,* which says that $(x^2 + y^2 + z^2)/3 \ge \sqrt[3]{(x^2\cdot y^2\cdot z^2)}$; it will be invoked in the equivalent form $|x\cdot y\cdot z| \le \|\mathbf{p}\|^3/\sqrt{27}$. Then substitution of trial arguments reveals that

$$f(-2\|\mathbf{p}\|/\sqrt{3}) < 0 \le f(-\|\mathbf{p}\|/\sqrt{3}) , \quad \text{and} \quad f(0) \le 0 \le f(2\|\mathbf{p}\|/\sqrt{3}) ,$$

so all three zeros of f (the eigenvalues of $|\mathbf{p}^{¢}|$) lie between $\pm 2\|\mathbf{p}\|/\sqrt{3}$. From this we infer that

$$\| \mathbf{w} - \mathbf{p}\times\mathbf{b} \| \le (4/\sqrt{3})\cdot\varepsilon\cdot\|\mathbf{p}\|\cdot\|\mathbf{b}\| .$$

A diagram consisting of a triangle establishes that $\| \mathbf{w} - \mathbf{p}\times\mathbf{b} \| \ge |\sin(\angle(\mathbf{w}, \mathbf{p}\times\mathbf{b}))|\cdot\|\mathbf{p}\times\mathbf{b}\|$; and we know that $\|\mathbf{p}\times\mathbf{b}\| = \|\mathbf{p}\|\cdot\|\mathbf{b}\|\cdot|\sin(\angle(\mathbf{p}, \mathbf{b}))|$. Assemble the last three relations to finish the proof that $|\sin(\angle(\mathbf{w}, \mathbf{p}\times\mathbf{b}))| \le \| \mathbf{w} - \mathbf{p}\times\mathbf{b} \|/\|\mathbf{p}\times\mathbf{b}\| \le (4/\sqrt{3})\cdot\varepsilon\cdot\|\mathbf{p}\|\cdot\|\mathbf{b}\|/\|\mathbf{p}\times\mathbf{b}\| = (4/\sqrt{3})\cdot\varepsilon/|\sin(\angle(\mathbf{p}, \mathbf{b}))|$.

Actually $4/\sqrt{3}$ can be replaced by $2/\sqrt{3}$ whenever $\sin(\angle(\mathbf{p}, \mathbf{b}))$ is very tiny since then all the elements of $\mathbf{p}\times\mathbf{b}$ are tiny because of cancellations during subtractions, which must then be exact, which removes one of the factors $(1\pm\varepsilon)$ from each element of $\mathbf{w}$.

Attempts to apply the last inequality falter when the unoriented angle $|\angle(\mathbf{p}, \mathbf{b})|$ differs from 0 or π by so little more than $\sqrt{\varepsilon}$ that its best-known textbook formulas are vitiated by roundoff:

The familiar formula $0 \le |\angle(\mathbf{p}, \mathbf{b})| := \arccos\left(\mathbf{p}^T\mathbf{b}/(\|\mathbf{p}\|\cdot\|\mathbf{b}\|) \right) \le \pi$ can lose to roundoff as many as half the significant digits carried by the arithmetic when $|\mathbf{p}^T\mathbf{b}|/(\|\mathbf{p}\|\cdot\|\mathbf{b}\|)$ differs from 1 by less than a few rounding errors. This happens because arccos skips so quickly at such arguments:

$$\arccos(1) = 0 ;\quad \arccos(1-\varepsilon) \approx \sqrt{2\varepsilon} ;\quad \arccos(1-2\varepsilon) \approx 2\sqrt{\varepsilon} ;\ \dots ;\quad \arccos(\varepsilon-1) \approx \pi - \sqrt{2\varepsilon} .$$

Consequently the arccos formula should be avoided when $|\angle(\mathbf{p}, \mathbf{b})|$ is near 0 or π .

Another formula almost as familiar is

$$\text{If } \mathbf{p}^T\mathbf{b} > 0 \text{ then } |\angle(\mathbf{p}, \mathbf{b})| = \arcsin\left(\|\mathbf{p}\times\mathbf{b}\|/(\|\mathbf{p}\|\cdot\|\mathbf{b}\|) \right)$$
$$\text{else } |\angle(\mathbf{p}, \mathbf{b})| = \pi - \arcsin\left(\|\mathbf{p}\times\mathbf{b}\|/(\|\mathbf{p}\|\cdot\|\mathbf{b}\|) \right) .$$

It can lose to roundoff as many as half the significant digits carried by the arithmetic when $\|\mathbf{p}\times\mathbf{b}\|/(\|\mathbf{p}\|\cdot\|\mathbf{b}\|)$ differs from 1 by less than a few rounding errors because, at such arguments, arcsin skips so quickly through angles near $\pi/2$. Avoid arcsin(…) for such angles.

More uniformly accurate than both familiar formulas is the following unfamiliar formula:

$$|\angle(\mathbf{p}, \mathbf{b})| = 2\cdot\arctan\left(\| \mathbf{p}/\|\mathbf{p}\| - \mathbf{b}/\|\mathbf{b}\| \| \,/\, \| \mathbf{p}/\|\mathbf{p}\| + \mathbf{b}/\|\mathbf{b}\| \| \right) .$$

Valid for Euclidean space of any dimension, it never errs by more than a modest multiple of ε .

(If the data's magnitudes are not so extreme that exponent over/underflow can invalidate it, the algebraically equivalent formula $|\angle(\mathbf{p}, \mathbf{b})| = 2\cdot\arctan\left(\| \mathbf{p}\cdot\|\mathbf{b}\| - \mathbf{b}\cdot\|\mathbf{p}\| \| \,/\, \| \mathbf{p}\cdot\|\mathbf{b}\| + \mathbf{b}\cdot\|\mathbf{p}\| \| \right)$ runs slightly faster.)